It would be the death of all science and the end of all progress if we could not even allow such laws as those of elementary arithmetic to count as truths. Nevertheless, even today Kronecker still has his followers who do not believe in the admissibility of *tertium non datur*: this is probably the crassest lack of faith that can be met with in the history of mankind.

However, a science like mathematics must not rely upon faith, however strong that faith might be; it has rather the duty to provide complete clarity. ([Hilbert, 1931b], pg. 268)

If I now believe a *deeper treatment of the problem* to be requisite, and if I attempt such a deeper treatment, this is done *not so much to fortify individual mathematical theories* as because, in my opinion, all previous investigations into the foundations of mathematics fail to show us a way of *formulating the questions concerning foundations so that an unambiguous answer must result.* But this is what I require: in mathematical matters there should be in principle no doubt; it should not be possible for half-truths or truths of fundamentally different sorts to exist. ([Hilbert, 1922c], pg. 198 italics added)

“The goal of finding a secure foundation of mathematics is also my own …”.

“…I should like to regain for mathematics the old reputation of *incontestable* truth, …”

“which [mathematics] appears to have lost as a result of the paradoxes of set theory” ([Hilbert, 1922c], pg. 200 italics added).

The antinomies of set theory are usually treated as border conflicts concerning only the most remote provinces of the mathematical realm, and in no way endangering the inner soundness and security of the realm and its proper core provinces. The statements on these disturbances of the peace that authoritative sources have given (with the intention to deny or to mediate) mostly do not have the character of a conviction born out of thoroughly investigated evidence that rests firmly on itself. Rather, they belong to the sort of one-half to three-quarters honest attempts of self-delusion that are so common in political and philosophical thought. Indeed, any sincere and honest reflection has to lead to the conclusion that these inadequacies in the border provinces of mathematics must be counted as symptoms. They reveal what is hidden by the outwardly shining and frictionless operation in the center: namely, *inner instability of the foundations on which the empire is constructed.* (Weyl [Weyl, 1921] pg. 86[1])

[1] Added italics follow Ewald's translation of Hilbert [Hilbert, 1922c], where this passage is quoted.

[O]ne sees that for the mathematician various methodological standpoints exist side by side. The standpoint that Weyl chooses and from which he exhibits his vicious circle is not at all one of these standpoints; instead it seems to me to be artificially concocted. ([Hilbert, 1922c], pg. 199)

Weyl justifies his peculiar standpoint by saying that it preserves the principle of constructivity, but in my opinion precisely because it ends with a circle he should have realized that his standpoint (and therefore the principle of constructivity as he conceives it and applies it) is not usable, that it blocks the path to analysis. ([Hilbert, 1922c], pg. 199)

"Thus we find ourselves in a great predicament: the most successful, most elegant, and most established modes of inference ought to be abandoned just because, from a specific standpoint, one has no grounds for them" ([Bernays, 1922b], pg. 218).

"no one, though he speak with the tongue of angels, could keep people from negating general statements, or from forming partial judgments, or from using *tertium non datur*" ([Hilbert, 1926])

"[t]he standpoints usually taken by mathematicians do not rest on the principle of constructivity at all, nor do they exhibit Weyl's circle" ([Hilbert, 1922c], pg. 199), Mathematicians have pursued to the uttermost the modes of inference that rest on the concept of sets of numbers, and not even the shadow of an inconsistency has appeared. If Weyl here sees an 'inner instability of the foundations on which the empire is constructed', and if he worries about 'the impending dissolution of the commonwealth of analysis', then he is seeing ghosts. Rather, despite the application of the boldest and most manifold combinations of the subtlest techniques, a *complete security of inference and a clear unanimity of results* reigns in analysis. We are therefore *justified* in assuming those axioms which are the basis of this security and agreement; *to dispute this justification would mean to take away in advance from all science the possibility of its functioning ...* ([Hilbert, 1922c], pg. 200 italics added)

Accordingly, a satisfactory conclusion to the research into these foundations can only ever be attained by the solution of the [mathematical] problem of the consistency of the axioms of analysis. If we can produce such a proof, then we can say that mathematical statements are in fact incontestable and ultimate truths–a piece of knowledge that (also because of its general philosophical character) is of the greatest significance for us. (Hilbert [Hilbert, 1922c], pg. 202)

The importance of our question about the consistency of the axioms is well recognized by philosophers, but in [the philosophical literature] I do not find anywhere a clear demand for the solution of the problem in the mathematical sense. ([Hilbert, 1922c], pg. 201)

[t]he great advantage of Hilbert's procedure rests precisely on the fact that the problems and difficulties that present themselves in the grounding of mathematics are transformed from the epistemologico-philosophical domain into the domain of what is properly mathematical. ([Bernays, 1922b], pg. 222)

"mathematics [thereby] takes over the role of that discipline which was earlier called *mathematical natural philosophy*" ([Bernays, 1931a], pg. 236).

Hilbert's theory does not exclude the possibility of a philosophical attitude that conceives of the numbers as existing, nonsensical objects [as Müller would have them be]. ...Nevertheless the aim of Hilbert's theory is to make such an attitude dispensable for the foundations of the exact sciences. ([Bernays, 1923], pg. 226)

We therefore see that, if we wish to give a *rigorous* grounding of mathematics, we are not entitled to adopt as logically unproblematic the usual modes of inference that we find in analysis. Rather, our task is precisely to discover *why* . . . we always obtain correct results from the application of transfinite modes of inference of the sort that occur in analysis and set theory. (Hilbert [Hilbert, 1923], pg. 1140 italics added)

One thus arrives at the attempt of a purely constructive development of arithmetic. And indeed the goal for mathematical thought is a very tempting one: Pure mathematics ought to construct its own edifice and not be dependent on the assumption of a certain system of things. ([Bernays, 1922b], pg. 217) . . .

For Hilbert in no way wants to abandon the constructive tendency that aims at the self-reliance of mathematics. ([Bernays, 1922b], pg. 219)

Accordingly, in Hilbert's theory we have to distinguish sharply between the formal image of the arithmetical statements and proofs as *object* of the theory, on the one hand, and the contentual thought about this formalism, as *content* of the theory, on the other hand. The formalization is done in such a way that formulas take the place of contentual mathematical statements, and a sequence of formulas, following each other according to certain rules, takes the place of an inference. And indeed no meaning is attached to the formulas; the formula does not count as the expression of a thought .... (Bernays [Bernays, 1922b], pg. 219)

"To reach our goal, we *must* make the proofs as such the object of our investigation; we are thus *compelled* to a sort of 'proof theory' which studies operations with the proofs themselves" ([Hilbert, 1922c], pg. 208 italics added).

Now the only question still remaining concerns the means by which this proof should be carried out. In principle this question is already decided. For our whole problem originates from the demand of taking only the concretely intuitive as a basis for mathematical considerations. Thus the matter is simply to realize which tools are at our disposal in the context of the concrete-intuitive mode of reflection. ([Bernays, 1922b], pg. 221)

[A] new sort of mathematical speculation [had] opened up by means of which one could consider the geometrical axioms from a higher standpoint. It immediately became apparent, however, that this mode of consideration had nothing to do with the question of the epistemic character of the axioms, which had, after all, formerly been considered as the only significant feature of the axiomatic method. Accordingly, the necessity of a clear separation between the mathematical and the epistemological problems of axiomatics ensued. ([Bernays, 1922a], pgs. 191-92)

"The demand for such a separation of the problems had already been stated with full rigor by Klein in his Erlangen Programme." ([Bernays, 1922a], pg. 92)

Mathematics has grown like a tree, which does not start at its tiniest rootlets and grow merely upward, but rather sends its roots deeper and deeper at the same time and rate as its branches and leaves are spreading upwards. Just so–if we may drop the figure of speech–mathematics began its development from a certain standpoint corresponding to normal human understanding and has progressed, from that point, according to the demands of science itself and of the then prevailing interests, now in the one direction toward new knowledge, now in the other through the study of fundamental principles. ([Klein, 1908])

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