Are all contradictions equal? Wittgenstein on confusion in mathematics

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Der Widerspruch.
Wieso gerade dieses *Eine* Gespenst?
Ludwig Wittgenstein

1 Introduction

In this paper I will not focus on contradictions as parts of formal languages (formulae in a logical system) or as an occasion to construct a system of paraconsistent logic, but I will rather study them as phenomena of mathematical practice. I shall do so from a Wittgensteinian point of view, using Wittgenstein's treatment of Cantor's Diagonal Argument as a guideline for most considerations in this paper.[1] There will be no main thesis resulting from my considerations, but I hope to be able to give a partial overview of what may happen, when a mathematician faces a contradiction. I will also discuss Wittgenstein's remarks on the mathematicians' attitudes towards the contradiction, but will leave aside the debate regarding the extent to which Wittgenstein should be understood as anti-revisionistic (cf. Frascolla, 1994; Maddy, 1992; Redecker, 2006; Wright, 1980, and many others).

> But you can't allow a contradiction to stand!—Why not? We do sometimes use this form in our talk, of course not often—but one could imagine a technique of language in which it was a regular instrument. (Wittgenstein, 1956, p. 166e)

Even if one does not call this revisionistic, one has to admit, I think, that adopting Wittgenstein's view would not leave things in mathematical practice as they are. (Of course, contradictions have always been a "regular instrument" in mathematics, in proofs by *reductio ad absurdum*, and also as means to develop and change theories (cf. Byers, 2007, p. 84 and p. 98)—but this is not: allowing a contradiction *to stand*.)

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[1]For a detailed discussion of Wittgenstein's remarks on the Diagonal Argument, cf. (Redecker, 2006).

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2 Contradictions which cause confusion (and those which do not)

If we are confronted with a contradiction (in a formal system), why do we not simply eliminate it by just excluding either the contradiction itself from the system or one of the propositions contributing to the contradiction?

> Can we say: 'Contradiction is harmless if it can be sealed off'? But what prevents us from sealing it off? That we do not know our way about in the calculus. Then *that* is the harm. And this is what one means when one says: the contradiction indicates that there is something wrong about our calculus. It is merely the (local) *symptom* of a sickness of the whole body. But the body is only sick when we do not know our way about.
>
> The calculus has a secret sickness, means: What we have got is, as it is, not a calculus, and *we do not know our way about*—i.e., cannot give a calculus which corresponds 'in essentials' to this simulacrum of a calculus, and only excludes what is wrong in it. (Wittgenstein, 1956, III, §80, p. 209)

Wittgenstein's answer is: Such a procedure will not be successful in cases where the contradiction makes us confused, where we cannot find our way ("wir kennen uns im Kalkül nicht aus").

But there are also other cases. Wittgenstein (1956, III, §80, p. 209) gives an example: One could teach Frege's calculus, which includes a contradiction. Nobody might notice it, and everybody would be content. Then: Where is the problem?

We need not even refer to such a thought experiment, mathematical practice provides us with a fairly recent "real life example": Physicists deal with what they call "δ-functions" and live in peace with the contradiction they produce.

This remarkable situation followed from several requirements in physics, especially in connection with solutions to certain differential equations. In theoretical physics, for example, the (ideal) situation of all mass concentrated in one point needs to be modelled. The *desideratum* would be a function f with the properties

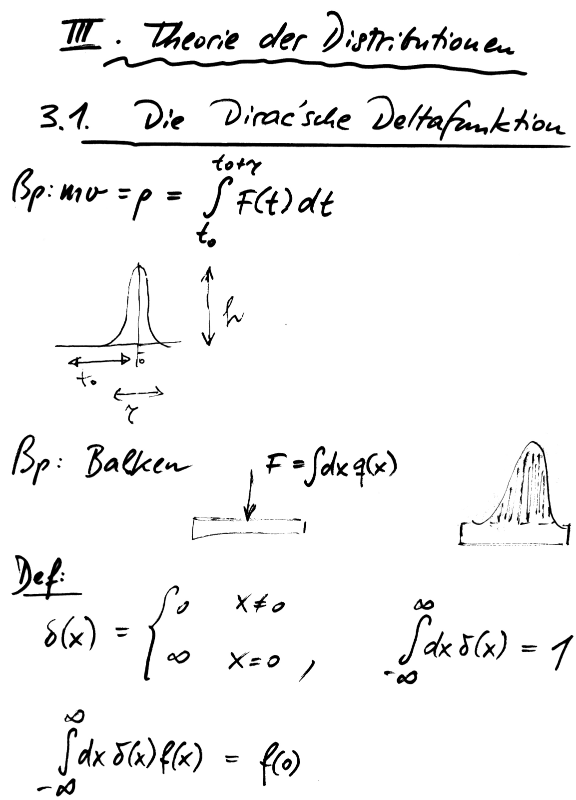
$$f(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ \infty \text{ (or } a \in \mathbb{R}) & \text{for } x = 0 \end{cases}, \quad \int\limits_{-\infty}^{\infty} f(x)\,dx = 1, \text{ and}$$

$$\int\limits_{-\infty}^{\infty} \delta(x) f(x) dx = f(0).$$

Such a function does not exist.

Mathematicians then built the theory of "generalized functions" or "distributions", i.e., linear functionals fulfilling the above requirements such that the "usual" functions can be embedded into the space of these functionals. But what the physicists really wanted were *functions*, not some generalized entities.

If you have a look at notes from a first term course in theoretical physics you will most likely still find something like this:



III. Theorie der Distributionen

3.1. Die Dirac'sche Deltafunktion

Bp: $mv = p = \int_{t_0}^{t_0+\tau} F(t)\,dt$

Bp: Balken $F = \int dx\, q(x)$

Def:

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}, \qquad \int_{-\infty}^{\infty} dx\, \delta(x) = 1$$

$$\int_{-\infty}^{\infty} dx\, \delta(x) f(x) = f(0)$$

This means that the physicists choose to stick to the self-contradictory objects. In a book on the history of the theory of distributions one finds:

> However, I have not drawn very general philosophical conclusions from the history told in this book, since I suspect the development of the theory of distributions may not be representative of the way mathematics has developed in the 20th century. (Lützen, 1982, p. 2)

Though distributions may not be a paradigmatic example for the history of mathematics in the 20th century, they are at least not un-typical for what becomes of mathematical concepts: From a later point of view we say that the mathematicians of earlier times did not really understand what they were doing,[2] that they did not avoid contradictions (because they did not notice them).

But in what sense did they really have a problem? Wittgenstein argues, that we only need to worry about a contradiction once we lose our understanding of what is going on.

3 Contradictions to start and those to end with

One of Wittgenstein's major concerns is Cantor's Diagonal Argument. A mathematician explaining the argument to a student might put it like this:

We may restrict ourselves to the real numbers between 0 and 1, for if these are uncountable, then the set of all real numbers will be so a fortiori. Now suppose the real numbers between 0 and 1 were countable, then we could make a list of all of them. *Let this list of real numbers, in their decimal notations, be:*

$$\begin{array}{l} 0, a_{11}a_{12}a_{13}\ldots \\ 0, a_{21}a_{22}a_{23}\ldots \\ 0, a_{31}a_{32}a_{33}\ldots \\ \vdots \end{array}$$

We can build a number between 0 and 1 in the following way: The first decimal place is $a_{11}+1$*, the second decimal place is* $a_{22}+1$*, the third is* $a_{33}+1$ *etc.—and we get a number, that is different from every number in the list, a contradiction. Therefore such a list does not exist; the real numbers are uncountable.*

Wittgenstein formulates two objections to the Diagonal Argument.

The first one is: We do not know what it means "to be a list of all real numbers (between 0 and 1)". The only conception of a list we possess is that of countable sets of things.[3] In Wittgenstein's words:

[2]Think of the convergence of infinite series, for example, or the infinitesimals of the early stages of the development of Calculus.

[3]And even in such cases there is reason for caution: "It is less misleading to say '$m = 2n$ allows the possibility of correlating every number with another' than to say '$m = 2n$ correlates all numbers with others'. But here too the grammar of the meaning of the expression 'possibility of correlation' has to be explained." (Wittgenstein, 1969, p. 466) (The English translation writes "time" for the German word "Zahl". We corrected this to "number".)

> Asked: "Can the real numbers be ordered in a series?" the conscientious answer might be: "For the time being I can't form any precise idea of that".—"But you can order the roots and the algebraic numbers for example in a series; so you surely understand the expression!"—To put it better, I *have got* certain analogous formations, which I call by the common name 'series'. But so far I haven't any certain bridge from these cases to that of 'all real numbers'. Nor have I any general method of trying whether such-and-such a set 'can be ordered in a series'. (Wittgenstein, 1956, II, §16, p. 130)

The second objection consists of the diagnosis that the outcome of the argument is a procedure to create a series of numbers rather than a thing of which we can be sure that it is a real number. Again, Wittgenstein:

> If someone says: "Shew me a number different from all these", and is given the diagonal rule for answer, why should he not say: "But I didn't mean it like that!"? What you have given me is a rule for the step-by-step construction of numbers that are successively different from each of these. (Wittgenstein, 1956, II, §5, p. 126)

Now both objections rely on this rather colloquial version of Cantor's argument. And Wittgenstein himself warns us:

> The result of a calculation expressed verbally is to be regarded with suspicion. The *calculation* illuminates the meaning of the expression in words. It is the *finer* instrument for determining the meaning. (Wittgenstein, 1956, II, §7, p. 127)

Such a "finer" alternative of the argument would be:

(We presuppose the real numbers to be given as Dedekind cuts or by some axioms.) Suppose φ *were a 1-1-mapping from* $\mathbb{N}$ *to [0,1]. For each* $b \in [0,1]$ *there is a decimal notation (this is a theorem), let's say* $0, b_1 b_2 b_3 \ldots$. *We define functions* $g_i : [0,1] \to [0,1]$ *such that*

$$g_i(b) = 0, \underbrace{0 \ldots 0}_{i-1} b_i 00 \ldots .$$

Then the series $\sum_{n=1}^{N}(g_n(\varphi(n)) + 10^{-n})$ *converges (for* N *to infinity) and therefore has as its limit a real number* d*, let's say (between 0 and 1). It can easily be shown that* d *is different from all numbers* $\varphi(n)$*,* $n = 1, 2, \ldots$. *This is a contradiction to the assumption that* φ *is one-to-one.*[4]

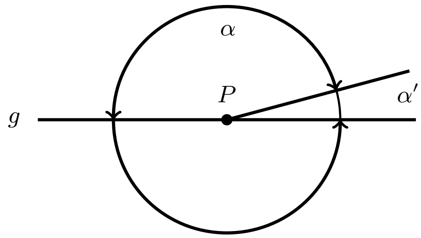
What remains of Wittgenstein's objections in the light of this more precise formulation?

[4]Instead of working with decimal expansions, this could be done with nested intervals. I have chosen this option because it is closer to Wittgenstein's considerations.

Let me start with the second objection. This objection vanishes with respect to the "mathematical" version; d is not "less a real number" than any number given as the limit of a sequence (as a Dedekind cut).[5] Not accepting d as a real number would mean to make Analysis impossible all together.

But in spite of his warning Wittgenstein continues to focus on the way we express things in ordinary language. We do in fact explain the Diagonal Argument as it was done in the first version (above), and Wittgenstein is quite right that the listener has not the slightest reason to identify this avoiding ("ausweichende") procedure (Wittgenstein, 1956, II, §8, p. 127) with a real number. The first version is simply not an appropriate way of telling the story. (It hides problems such as that of totality for instance.)

The first objection is more serious with regard to the formal version. Applied to this version it says: we do not know what a one-to-one-function between a countable and an uncountable set "looks like", we do not know any examples. Is this just an objection to any sort of *reductio ad absurdum*?

> How does indirect proof work, for instance in geometry? What is strangest about it is that one sometimes tries tries to draw an ungeometric figure (the exact analogue to an illogical proposition).
>
> 
>
>
> (But this, of course, only comes from an erroneous interpretation of the proof. It is funny, for instance, to say "assume that the straight line g has two continuations from point P". But there is really no need to assume such a thing.)
>
> Proofs in geometry, in mathematics, cannot be indirect in the real sense of the word because one cannot assume the opposite of a geometrical proposition as long as one sticks to one specific geometry.
>
> (That proof simply shows that the arcs α and $\alpha+\alpha'$ approximate each other all the more and without limit, the more α' approximates 0.) (Wittgenstein, 2000, Item 108, p. 29, transl. E. R.)

[5] Wittgenstein could argue that the limits of sequences or series are no more numbers than the result of the diagonal proof, and indeed he is inclined to say that the definition of a limit includes a proof (Wittgenstein, 1956, V, §36, p. 290), but this will not keep us from regarding the result as a number; otherwise the answer to a question could never be a number because a question is never a number.

Obviously Wittgenstein really does not accept indirect proofs, at least not in every case. On the other hand there are remarks like:

> What an indirect proof says, however, is: "If you want *this* then you cannot assume *that*: for only the opposite of what you do not want to abandon would be combinable with *that*." (Wittgenstein, 1956, p. 147e)

This sounds as if Wittgenstein were, in principle, in agreement with the idea of the *reductio ad absurdum*. Consider the following examples:

Example 1.

> **Theorem.** $\sqrt{2}$ is not a rational number.
>
> *Proof.* Suppose $\sqrt{2}$ were rational, then it could be written as $\frac{a}{b}$ ... Q.E.D.

Example 2.

> **Theorem.** Every sequence (x_n) in a subset F of $\mathbb{R}$ has a convergent subsequence in F if and only if F is bounded and closed.
>
> *Proof (one direction).* Consider a subset F of $\mathbb{R}$. Suppose it is not closed. Choose x in the closure of F, but not in F, then for all $n \in \mathbb{N}$ there is an $x_n \in F$ such that $|x_n - x| < \frac{1}{n}$. Then every subsequence of (x_n) converges to x, but $x \notin F$, and therefore (x_n) has no convergent subsequence *in* F—a contradiction to the assumption. Q.E.D.

Obviously the first proof—which is similar in this respect to Cantor's Diagonal Argument—starts with the contradiction, whereas the second proof just uses the contradiction in the end.

We could conclude that Wittgenstein does not accept those proofs by *reductio*, which start with a (hidden) contradiction—but that seems too superficial a distinction. The following section will describe what could be seen as the "deeper" (Wittgenstein does not like this word) distinction behind the "at the beginning/at the end"-distinction.

I agree that the latter distinction may be regarded as superficial, but want to point out that Wittgenstein's understanding of proofs as being fundamentally dependent on their geometry (the "geometry of signs" as he calls it)[6] makes it important *where* the contradiction is placed.

[6]The role of visual thinking in symbol manipulation is studied extensively in (Giaquinto, 2007, pp. 191–213)—not from a Wittgensteinian, but from a cognitivistic point of view; nevertheless there are similarities. Cf. also (Rotman, 2000, pp. 44–70).

4 Contradictions which include the impossibility of an intuition or a concept (and those which do not)

In the first example we suppose something that contradicts itself and of which we therefore cannot have any concept or imagination. The Diagonal Argument is similar to the first example in this respect. We do not know what a "list of irrationals" should be—just as we do not know what a "list of water" should be.[7] In the second example everything works well and is coherent *as long as the presupposition is not taken into consideration*. (Notice that we can start the proof with "suppose it *is*" instead of "suppose it *were*".) We do not have to start with a contradiction but may decide in the end "if you want *this*—F has no sequences without convergent subsequences—, then you must not assume *that*—F is not closed".[8]

How to deal with the Diagonal Argument now? We might admit that the proof is nonsense—because we started with an absurdity—and therefore give up the idea of proving (in this manner) anything about the size of $\mathbb{R}$ in comparison with $\mathbb{N}$. What we have learnt then is just:

> If it were said: "Consideration of the diagonal procedure shews you that the *concept* 'real number' has much less analogy with the concept 'cardinal number' than we, being mislead by certain analogies, are inclined to believe", that would have a good and honest sense. But just the *opposite* happens: one pretends to compare the 'set' of real numbers in magnitude with that of cardinal numbers. The difference in kind between the two conceptions is represented, by a skew form of expression, as difference of expansion. (Wittgenstein, 1956, II, §22, p. 132)

We do not know how to compare them by listing, one-to-one-mapping, etc. And the fact that we do not know this tells us, that they are different. And the proof might be taken as a hint that there is something wrong with any attempt to compare their sizes the way we tried to. So the contradiction destroys parts of the supposed meaning of a concept, it does not render a proposition false.

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[7] This is, I suppose, a case where the contradiction leads to confusion in the sense of the first section.

[8] Of course one might argue that the proof of Example 1 can easily be restated in the same form: Let x be a rational number, $x = \frac{a}{b}$, etc., and in the end it turns out, that $\sqrt{2}$ would not be a possible candidate for such a number. But this is not how we *do it*! Wittgenstein is right that there are two different ways of proceeding in a proof by *reductio in practice*. In Example 1 we work with the expression $\sqrt{2} = \frac{a}{b}$ throughout the whole proof—and this is clearly a meaningless expression as we very well know from the onset.

5 Contradictions between mathematical propositions and contradictions between mathematical and other propositions (empirical ones)

In the early years after his return to philosophy, Wittgenstein thought that the meaning of a mathematical proposition consisted in its proof—at least this seems to be the direction he inclined to. Later on he places greater emphasis on the relation between mathematical propositions and their "roots" in everyday life. But throughout his life the discussion of the status of (certain) propositions is at the centre of his remarks on mathematics. So the topic of the current section would require a detailed analysis of "logical", "grammatical", "empirical" propositions and the propositions called "hinge propositions" in secondary literature. As this goes far beyond the scope of this article I will restrict myself to one little remark.

> Put two apples on a bare table, see that no one comes near them and nothing shakes the table; now put another two apples on the table; now count the apples that are there. You have made an experiment; the result of counting is probably 4. [...] And analogous experiments can be carried out, with the same result, with all kinds of solid bodies.—This is how our children learn sums; for one makes them put down three beans and then another three beans and then count what is there. If the result at one time were 5, at another 7 (say because, *as we would now say*, one sometimes got added, and one sometimes vanished of itself), then the first thing we said would be that beans were no good for teaching sums. But if the same thing happened with sticks, fingers, lines and most other things, that would be the end of all sums.
>
> "But shouldn't we then still have 2+2=4?"—This sentence would have become unusable. (Wittgenstein, 1956, I, §37, p. 51f)

This passage has a harmless reading: We place more trust in mathematics (mathematical propositions) than in our ability to count (correctly).[9] (And only if mathematics were not expedient in the majority of cases, would we consider mathematical propositions to be meaningless.) But there is another reading: We would exclude beans from the teaching of elementary arithmetic if they did not behave well and disappeared—this might still seem harmless, but note that this implicitly states that we would rather believe that a bean has dissolved into nothing than that 3+3 is not 6. So it is not only our possible errors that make us distrust—if we take Wittgenstein seriously in what he says here, we would rather adopt the assumption of a very incoherent "world of experiences" than give up our acceptance of arithmetical theorems.

[9] And this is the "main reading" of the passage. I agree with Gierlinger (2008, p. 123) in this point.

At first sight this is not only astonishing *per se*, it also seems to contradict what I presented as Wittgenstein's attitudes in the earlier sections of this article. If mathematicians overestimate the general importance of contradictions, why should we then adopt the deeply unsettling assumption that things can simply disappear, only because 3+3=5 contradicts 3+3=6?

It is not just *any* contradiction we have to deal with here, it is not just the two mathematical propositions contradicting each other, it is the whole system of elementary arithmetic together with all our practices of counting, calculating, paying bills, etc. that is at stake. This situation could be taken as a paradigm for what is meant by a contradiction that causes confusion.

But is not the confusion, that arises from the assumption that things might disappear, even worse? I think the solution of the problem consists in recognizing that we have the wrong picture of what is at issue here. The situation does not, as it might seem, resemble a scale with elementary arithmetic on the one side and the trustworthiness of our daily experience on the other. First, the possibility for things to dissolve is not Wittgenstein's interest in the quoted passage, so we might perhaps better take take his statement *cum grano salis*. And second, in the situation he describes, *only* beans dissolve (if all things can disappear, then mathematics, too, becomes useless—but this would be taking matters one step further). So Wittgenstein contrasts a very local phenomenon with a very global one (arithmetic). And pressed to choose between the two, we would decide in favour of the global regularity.[10]

6 Contradictions which we seek and those which surprise us

This distinction does not coincide with a distinction between "contradictions in indirect proofs" and "real" contradictions (those which "harm" the mathematicians). In practice something can *turn out* to be an indirect proof. You try to prove something and suddenly you realize that you have found a contradiction. So "contradictions of indirect proofs" are not a subspecies of "contradictions we seek".

The comparison I have in mind here is rather between contradictions we are suddenly confronted with, ones that shock us, and those we search for. Of course (some of the) indirect proofs are examples for the second sort. Another example from Wittgenstein:

> For might we not possibly have *wanted* to produce a contradiction? Have said—with pride in a mathematical discovery: "Look, this is how we produce a contradiction"? Might not e.g. a lot of people

[10]I nevertheless perceive Wittgenstein's argument as problematic: He wants us to imagine a world where beans have the ability to dissolve, but nothing else has. We have to regard this as a thought experiment and be very charitable to engage in it.

> possibly have tried to produce a contradiction in the domain of logic [...]? These people would then never actually employ expressions of the form $f(f)$, but still would be glad to lead their lives in the *neighbourhood* of a contradiction. (Wittgenstein, 1956, III, §81, p. 211)

There is something curious about the first sort of contradictions: Wittgenstein once states that "if you are surprised, then you have not understood it yet".[11] Applied to contradictions this means: if the contradiction surprises you, then you do not understand what you do not understand. On the other hand it is quite natural to see a contradiction as the point at which it becomes manifest that you do not understand what is going on. So if the result of a proof surprises you, you are expected to think it through again, until the surprise gives way to understanding. But what if you are surprised by a contradiction? Should you try to understand the contradiction better? This does not seem to be very sound or reasonable. I will not follow this line of thinking here, but there is any case a seamless transition from trying to understand the steps that led to a contradiction to searching for a contradiction.

Now we are in the very centre of Wittgenstein's objections against the mathematicians' exaggerated fear of contradictions. He doubts that it is of any use to try to avoid contradictions mechanically in cases where we do not have any reason to distrust the system we are using (cf. Wittgenstein, 1956, III, §83, pp. 214ff).

The working experience of mathematicians gives them no motive to suppose that there could be a contradiction in elementary arithmetic and therefore—according to Wittgenstein—no reason to busy themselves with proofs of soundness of this system. (Note that this is exactly what mathematicians nowadays do—and did when Wittgenstein wrote his remarks, which was just after Gödel had completed his paper and the foundations of mathematics constituted a central topic—, however, they are doing so for pragmatic reasons, whereas Wittgenstein offers a philosophical background for this attitude vis-à-vis contradictions.)

7 Concluding remarks

Wittgenstein wonders why it is just "this *one* bogy" (Wittgenstein, 1956, IV, §56, p. 254)—the contradiction—that mathematicians seem to fear so much. An answer could be given by following his own considerations: There is not just one kind of contradiction, there are many. And they can hamper the mathematician's work in many different ways. But this does of course not answer the question Wittgenstein actually had in mind: Why no *other* bogies? And again we find hints for an answer in the very passages I have

[11] (Wittgenstein, 1956, App. II, §2, p. 111); for Wittgenstein on surprise in mathematics cf. (Floyd, 2008, 2010; Mühlhölzer, 2001).

discussed: Many different threats become visible when we analyse what happens when contradictions occur. For example the contradiction we face when we try to compare the natural with the real numbers with respect to their magnitude teaches us to be careful with expansions of concepts to other areas. A much simpler example: There is nothing to be said against talking of the "length" of a desk and also of the "length" of the distance between the earth and the moon—but we must not forget that we measure them in totally different ways and that "length" therefore has a different meaning in the two cases (Wittgenstein, 1956, III, §4, p. 147). The same holds for expansions of concepts in mathematics in general. Wittgenstein also discusses the transition from finite numbers to infinite cardinal numbers. He argues that a proof concerning finite numbers consists in ascertaining that it holds for every number of the finite totality. Whereas—as there is no infinite totality—a proof for all numbers has to yield a procedure. Stuart Shanker sums up:[12].

> The word 'class' means *totality* when it is used in the context of a finite *Beweissystem* (a group of objects all sharing the same property); but in its 'infinite' framework 'class' signifies a rule-governed series (the possibility of constructing a series *ad infinitum* by the reiteration of an operation)." (Shanker, 1987, p. 165)

So thoughtless expansion of concepts could be considered another bogy, we should beware of.

As we have seen, a contradiction may be, but need not be a symptom of a state of confusion. *If* it is a symptom of a state of confusion, then it can be seen within a more general context: Wittgenstein says, "a philosophical problem has the form: I don't know my way about" (Wittgenstein, 1953, § 123). And this is exactly what a contradiction in the "serious" sense says. The contradiction then tells us that we do not know our way about, but it *only* tells us *that*. It is the mere expression of perplexity; it tells us where the problem is, not what the problem is. If we want to understand the reason for the perplexity—if we want to see the philosophical problem—, we must have a look around, investigate the context of the contradiction. My proposal to see a confusing contradiction in a more general context can therefore supplemented by the "opposite" statement: A more specific context (a philosophical tradition, for example) is needed to understand what has happened.

As I said at the beginning of this text, I will not state a thesis, but I am now able to formulate a suggestion, which is not entirely unfounded:

[12]Wittgenstein's interest in "system", "class" and "totality" has one of its origins in Waismann's considerations and discussions with him; cf. (Waismann, 1986).

it could be useful to search for "other bogies" in the neighbourhood of contradictions.[13]

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[13] I would like to thank the referee for very helpful remarks on this concluding section.

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