Proof: Some notes on a phenomenon between freedom and enforcement

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1 Introductory remarks and a disclaimer

Proof lies at the core of mathematics. A large part of every day's work in mathematical research is devoted to searching for provable theorems and their proofs, explaining proofs to students or colleagues, and reading other mathematician's proofs and trying to understand them. Mathematics is special among the other sciences due to the role of proof and insofar as mathematical proof offers the highest possible rigor. Moreover, it is a special feature of mathematics—in contrast to all positive sciences—that it can work on its foundations using its own methods. The mathematical analysis of mathematical proof is systematically developed in 20th century's formal logic and (mathematical) proof theory.[1] Here it was possible to transform proof from the *process* of mathematical arguing to an *object*—e.g., a well-formed sequence of symbols of some formal language—mathematics can argue about. Though this approach offers valuable results I will not concentrate on it.

In fact, there has been a growing interest in 'non-formal' features of mathematics. In fact, a closer look at the 'real existing' mathematics exhibits a much less formal science than the formalistic picture claimed. Here historical studies,[2] arguments from 'working mathematicians',[3] and sociological studies[4] point into a similar direction. Concerning a non-formal 'phenomenology of proof' we refer to the recent discussion initiated by Yehuda

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[1] In his comprehensive work on the philosophy of mathematical proof theory, Wille (2008) emphasizes that proof theory is special among the mathematical subdisciplines because it discusses *normative* questions about external justifications of its axiomatic basis. In fact, it can not be denied that the content of the *publications* in proof theory vary between a purely mathematical and a less technical one comprising normative claims and argumentations. However, in *every* mathematical area there are normative discussions—however rarely published—about the 'sense' of axioms, definitions, the fruitfulness of results etc. The only difference seems to be the communication medium.

[2] Cf., e.g., the pioneering work of Imre Lakatos (1976).

[3] Cf., e.g., the polemic in (Davis and Hersh, 1981) against a philosophy of mathematics emphasizing only the formal aspects.

[4] Cf., e.g., (Heintz, 2000).

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Rav (1999).[5] With reference to many concrete examples from the mathematical practice it is argued there, that most of the existing proofs are for good reasons non-formal. Only in special situations strictly formalized proofs, called *derivations* occur. We shall contrast this with the classical analytic-synthetic distinction below. A special emphasis is given on the heuristic value of (the semantic side of) proof for the scientific practice. In the related paper (Pelc, 2009), the focus is narrower, concerned only with the function of proof to foster confidence for a claimed theorem. Nevertheless, the importance of non-formal proofs is underpinned, since a transformation of interesting proofs into derivations is hopeless for complexity reasons.

In our note we shall follow a similar, but slightly modified question: 'why and how are we convinced by proofs?' In fact, we are not so much interested in the important question of the genesis and growth of mathematics, but into the question, *how* the validation of its results work. This note, however, does not intend to bring forward any philosophical argument and avoids thus the debates and counter-debates of the professional philosophy of mathematics. Instead, we shall just present a collection of observations from a practitioner's point of view.[6]

2 Brian Rotman's semiotic perspective on mathematics

This note will adopt a perspective which is influenced by semiotics;[7] we thus observe proof as a quite special communication process[8] between two or more people or inside one person. Brian Rotman (1988) has offered this semiotic perspective on mathematics in an inspiring paper. Here he presents an integrative view where the basic insights of intuitionism, formalism, and platonism are valuated but also their respective limitations are shown:[9]

> [...] to have persisted so long each must encapsulate, however partially, an important facet of what is felt to be intrinsic to mathematical activity.

For a closer description of the working mathematician's dealing with signs, Rotman splits him into three different acting entities, the *Agent*, the *Mathematician*, and the *Person*:

> If the Agent is a truncated and abstracted image of the Mathematician, then the latter is himself a reduced and abstracted version of

[5]Cf. the subsequent discussion in (Azzouni, 2004; Pelc, 2009; Rav, 2007).

[6]Compare the introductory remarks in (Rotman, 1988, p. 97).

[7]A related perspective on proof is presented in (Lolli, 2005).

[8]I will concentrate, however, still on proof as an idealized concept of communication. For sociological observations of the 'real existing' mathematical communications, cf., e.g., (Heintz, 2000) or, more recently, (Wilhelmus, 2008).

[9]Cf. (Rotman, 1988, p. 101).

> the subject – let us call him the Person who operates with the signs of natural language and can answer to the agency named by the 'I' of ordinary nonmathematical discourse. [...] The Mathematician's psychology, in other words, is transcultural and disembodied.

While jointly doing mathematics the Mathematician and the Agent perform different types of activities, e.g.,

> [...] it is the Mathematician who carries out inclusive demands to 'consider' and 'define' certain worlds and to 'prove' theorems in relation to these, and it is his Agent who executes the actions within such fabricated worlds, such as 'count', 'integrate', and so on, demanded by exclusive imperatives.

Solely the Mathematician can communicate the leading idea of a proof to another Mathematician, while many activities the Mathematician is incapable to perform, e.g., evaluating infinite series, are done by the Agent. Only the combined activities of these two give a full picture.

We shall follow this perspective and try to emphasize one aspect, Rotman only implicitly touches: The question of freedom and enforcement. Our leitmotif will thus be, that mathematical proof shows a strange tension of *freedom* and *enforcement* and can only be understood within this field. It is thus our aim to pinpoint some characteristic features of mathematical proofs where this tension can be observed. With respect to Rotman's actors, we might characterize the Agent to be completely determined by the orders of the Mathematician, who in turn is at least (partially) free to define certain objects, to suppose certain conditions etc., but restricted to the rules of the mathematical discourse. The Person, finally, is (completely) free to join or to leave this discourse.

3 Proof: despotism without authorities

One obvious goal of a mathematical proof, though not the only one, is to give evidence that a certain claim is true. But, how does this takes place? To characterize an important feature of the pragmatics[10] of proof I would like to recall an instructive example: the elementary proof of Pythagoras' theorem in Plato's dialogue *Menon*. Here, I will not summarize the content of this wellknown dialogue, but just concentrate on some points. The main subject is a foundational question of ethics namely for the nature of virtue and whether it could be taught and learned. This question leads to the

[10] A completely different pragmatic aspect of proof is the concern of Imre Lakatos (1976) in his classical work. Here it is shown that giving evidence for the truth of a result is only *one*—and often not even the most important—goal of a proof. It is much more important that the argumentation gives hints for understanding the respective roles of the assumptions and thus for understanding *why* the theorem could hold.

general dilemma of teaching and learning: It seems impossible to learn anything unknown. If we do not know a content, after finding it we can never realize that it was the previously unknown. Socrates solves the puzzle by the concept of *anamnesis*: learning is nothing else than remembering the ideas already seen by the soul before coming to earth. To give evidence for this claim he demonstrates Pythagoras' theorem to an uneducated slave and *at the same time* he 'demonstrates' to Menon, how something previously unknown could be learned. The *aha effect* at the end of the learning process is afterwards interpreted as remembering. After knowing the solution it seems as if it was always known; the question—unsolvable before the proof—is now trivial. The initial state of ignorance appears to be improper, properly spoken, the theorem was always known. Indeed, the phenomenology of *learning* in mathematics and *remembering* is stupendously similar. For the initial question, whether virtue could be learned, Socrates claims at least the chance for a positive answer. Though there is no guaranteed way or indoctrination, we should steadily undertake efforts and hope for finding virtue by divine grace.

It is, however, an ironical feature of the dialogue, that the 'hard' argument for this position is a mathematical proof. The result is thus fixed from the outset and during the course of the proof every alternative route led into obvious errors. So the slave had to follow the necessitating force of proof.[11] There is no fair bargaining about the result, and at the end no freely chosen assent to the theorem; the communication process has thus a **dictatorial** character. On the other hand, the social position or any authority of the dialogue partners is completely irrelevant; thus there is also a **subversive** aspect, the freedom of arguing against any authority. Instead of a dictatorial proclamation of the result mathematical proof argues for it. David Hilbert characterized the value of mathematics for general education mainly in "ethical direction" since it "awakes the confidence in our own intellect, the critical ability of judgment, which distinguishes the truly educated person from the one who merely beliefs in authorities."[12] However, proof is based on strict commands formulated in the definitions and axioms and following the argument of a proof means following the commands of the author; Herbert Mehrtens felicitously describes this basis:

> Die Setzungen der Mathematik haben den Charakter von Befehlen, die Theoreme und Schlußfolgerungen sollen immer zwingende Folge der Befehlssysteme sein. Das macht den eigenartigen Charakter dieser

[11] The winning of dialogues is consequently the intuitive background for a constructive approach to formal logic by Paul Lorenzen and Oswald Schwemmer. A claim is provable, if there exists a definite strategy for winning such a dialogue, cf. (Lorenzen and Schwemmer, 1973).

[12] Cf. Hilbert's 1922/23 lectures edited in (Bödigheimer, 1988, p. 3; translation by the author).

> Sprache aus; sie besteht aus Befehlen, die das Setzen von Zeichen regeln. Die Gewißheit der Mathematik liegt in ihrer befehlsmäßig zwingenden Struktur.[13]

Taking stock, we regard a strange tension in the communication process. On the one hand, a mathematical proof is an ideal communication, which is strictly *herrschaftsfrei* (free of dominion) in the sense of Jürgen Habermas. On the other hand, there is absolutely no tolerance for defending a differing position. Only the predetermined result can be defended with warranty.

4 Freely chosen objects—strictly ruled language

There seems to be only one way for escaping the subversive despotism of a mathematical proof—aside from refusing to listen at all. For successfully denying the theorem you could only change the axioms, e.g., switch from plane to spherical geometry. This leads to the question: what are we talking about in mathematical proofs? During the course of the early 20th century we can observe a major change with respect to this question. It is well-known that the result is a switch from external to almost purely internal reference leading to a far reaching autonomy of mathematics. It lies in the *free* choice of the mathematician, which special set of axioms he likes to start with.[14] No external object dictates a certain set. And it is hardly exaggerated when Georg Cantor claims “Das Wesen der Mathematik liegt in ihrer Freiheit” (Purkert and Ilgauds, 1987, p. 121).

Concerning the communication process, we observe that conflicting positions in mathematics can now simply be avoided by subdiscipline branching; if you don't like Euclidean geometry you just change to hyperbolic. If you don't like the axiom of choice you work without it. This is similar in proof theory: you may freely choose from various axiomatic means. This also seems to be *one* reason, why mathematicians will quite fast agree about the validity of an argument.

However, this freedom is restricted in a twofold way. First, there is no freedom of interpretation and no context dependence of the terms. In contrast to all other texts, within a mathematical proof every x must strictly remain the same x. There is—so to speak—no hermeneutical problem in a mathematical proof. Of course, this does neither mean that every proof

[13] “The determinations of mathematics have the character of commands, the theorems and inferences must be coercive implications of the systems of commands. This constitutes the peculiar character of this language; it is composed of commands ruling the drawing of signs” (translation by the author), cf. (Mehrtens, 1993, p. 101).

[14] Herbert Mehrtens discusses the development of modern mathematics and the disputes during the ‘foundational crisis’ just under this aspect of creative freedom, cf. (Mehrtens, 1990). Here David Hilbert—following Cantor—stands for a progressive modernity against L. E. J. Brouwer—following Leopold Kronecker—being the representative of reactionary anti-modernity which claims a necessary external reference for mathematics.

is self evident to everybody nor that the inner pictures connected to x are the same for everybody. This strong concept of identity enables and leads to the second restriction: the chosen axioms are not allowed to contain contradictions neither explicitly nor implicitly. The anxious emphasis on this consistency, is the prize we pay for the freedom of choice with respect to the axioms.

5 Proof between determined calculation and spontaneous construction

If we cease to refer to given objects 'somewhere outside' there is no other way than to look at the internal structure of mathematical proof to explain its force.

To characterize mathematical inferences I will refer to the classical distinction of *analytic* and *synthetic* judgments[15] due to Immanuel Kant.[16] Though it is disputed, whether this distinction can be applied to all judgments, I think that it still can be used to illustrate important features of mathematical proofs. Thereby, I will postpone the question, whether mathematical proofs *are* empirically based or *a priori*. It is simply observed that mathematicians claim necessity for their inferences, thus try to avoid any empirical flavor. Moreover, the analytic-synthetic distinction can be used independently form the decision about the *a priori* character.

On the one hand, a correct proof can be described as consisting in or being transformable into a—probably very long—chain of identities. This transformation is either a real goal or at least an ideal. Any theorem is thus implicitly contained in the axioms; every judgment is analytic. Here, the assent enforcing power of proof seems to be well captured since nobody could reasonably deny the identity of the identical. We find this position most clearly stated in Leibniz's writings. In his small paper "Zur allgemeinen Charakteristik" he sketches the picture of proof—and broadens the application to all human reasoning—being as simple as a mechanical calculation:

> [E]s müßte sich, meinte ich, eine Art Alphabet der menschlichen Gedanken ersinnen und durch die Verknüpfung seiner Buchstaben und die Analysis der Worte, die sich aus ihnen zusammensetzen, alles andere entdecken und beurteilen lassen [...] Unsere Carakteristik wird alle Fragen insgesamt auf Zahlen reduzieren und so eine Art

[15]The usefulness of this distinction has been doubted, e.g., in (Quine, 1951). However, Quine's identification of analytic with "grounded in meanings independently of matters of fact" and synthetic with "grounded in fact" is for our concern much too close to the a priori vs. a posteriori distinction.

[16]It is not our aim to give an interpretation of Kant's complex philosophy of mathematics; for a concise analysis in the context of its functions within his critical philosophy, cf. (von Wolff-Metternich, 1995).

> Statik darstellen, vermöge derer die Vernunftgründe gewogen werden können.[17]

Almost the same characterization of a quasi mechanical process of inference without any thinking can be found in Paul Bernays' writings:

> [N]achdem einmal die Prinzipien des Schließens genannt sind, [braucht] nichts mehr überlegt zu werden. Die Regeln des Schließens müssen so beschaffen sein, daß sie das logische *Denken* [emph. G.N.] eliminieren. Andernfalls müßten wir ja erst wieder logische Regeln dafür haben, wie jene Regeln anzuwenden sind. Dieser Forderung der Austreibung des Geistes kann nun wirklich genügt werden.[18]

There are—according to Leibniz—two basic principles for these mechanical proofs,

> die Definitionen oder Ideen und [...] ursprüngliche, d.h. identische Sätze, wie der, daß B gleich B ist.[19]

If we put aside the external reference of the axioms, all mathematical theorems and proofs reduce to identities. There are, however, at least two more drawbacks for this position. The first one on the 'macro-level': no interesting proof has ever been completely broken down to trivialities. Nobody would be able to write or to read and *understand* it.[20] To overcome this problem we could theoretically refer to 'in principle analysis' or to the metaphor of a 'mechanical working' of proof and practically involve computers. In fact, Leibniz's characterization, the concept of analyticity is well illustrated by the mechanical working of a machine. Being just a metaphor for Leibniz, today *computer (assisted) proofs* are technically implemented and became an interesting tool for mathematical research. The status, however, of these proofs is still a question in dispute. Of course, a first question arises, how we shall guarantee its correctness, thus the correct working of the software *and* the hardware of the computer. However, McEvoy (2008) argues against others (e.g., Kitcher (1983) or Resnik (1997)) that computer

[17] "I arrived at this remarkable thought, namely that a kind of alphabet of human thoughts can be worked out and that everything can be discovered and judged by a comparison of the letters of this alphabet and an analysis of the words made from them. [...] Our characteristic will reduce all questions to numbers, so that reasons can be weighed, as if by a kind of statics" (translation by the author), quoted after (Cassirer, 1966, pp. 30).

[18] "After designation of the principles of inference, no more thinking is needed. The rules of inference must eliminate the logical thinking. Otherwise we would need new logical rules how these rules can be applied. This demand of an expulsion of spirit can actually be satisfied" (translation by the author), cf. (Bernays, 1976, pp. 9).

[19] "[...] definitions or ideas and [...] identical propositions such as $B = B$" (translation by the author), cf. (Cassirer, 1966, p. 58).

[20] This is the core thesis in (Pelc, 2009).

assisted proofs are as *a priori* as any other 'hand-made' proof. In fact, the checks and double checks of a proof can always be called 'experiments', independently on the question, whether it is a machine or a human mathematician who performs it. And in both cases we shall never be absolutely certain if the proof is sufficiently long. Accepting this view, the grade of warrant and the aprioricity is therefore not the main difference between a hand-made and a computer proof. Instead of the *grade* of certainty, it is rather its *base*, we should look at. A human mathematician performs an analytical inference by a (freely chosen, but) trusted assent following the self-posed rule, e.g., of *modus ponens*. The working of a machine, on the other hand, obtains its reliability by a supposed strict determination. We thereby replace *necessity* of the result by a *determined contingency*. And at least on the theoretical level, we must suppose a strong concept of determinism. It is not by chance—I think—that the same Leibniz draws one of the strongest metaphysical pictures of a completely predetermined course of the world. It is, however, to question whether we do want to refer to such a strong metaphysical concept when trying to understand the concept of a mathematical proof. Second, even on the 'micro-level' there still remains a question. If we accept a proof to be a chain of tautologies, we should ask again why or how such a single tautology—thus the principle of identity — convinces. I do not want to deepen this question here. It may be sufficient to mention that, e.g., the sociologist Niklas Luhmann[21] again and again considers this and states a puzzling paradox of identity. Another reference is Klaus Heinrich (1981) who studies the origin of basic logical principles in the Greek myth. Both authors show that the seemingly trivial tautologies are quite complicated phenomena.

On the other hand, it seems strongly that a successful mathematical proof—as a whole and in every single step—is a non-trivial act of constructive unification. Its basis lies within a spontaneous, non-mechanical, constructive working by the mind of the (human) mathematician. It is Kant who emphasizes the role of pure intuition,[22] *reine Anschauung*, and the synthetic character of mathematical theorems and proofs.[23] First, Kant takes the axioms[24] to be synthetic and *a priori* valid propositions which

[21]Cf., e.g., (Luhmann, 1992, pp. 491). Here Luhmann formulates "die Einsicht, daß die Tautologie letztlich nichts anderes ist als eine verdeckte Paradoxie; denn sie behauptet einen Unterschied, von dem sie zugleich behauptet, daß er keiner ist."

[22]The emphasis on intuition can also be found in the *Menon* dialogue. The switch from arithmetic to geometry is precisely the point where intuition is essentially required. It is important to remark here that for the Greek mathematics there was no arithmetical solution of the problem, no calculus dealing with irrational 'numbers' as, e.g., $\sqrt{8}$. It was thus inevitable to switch from the arithmetical calculation to the intuitive geometry.

[23]In (Norman, 2006), this synthetic character is further substantiated when the author discusses the role diagrams play for mathematical reasoning.

[24]Cf. (Kant, 1781, pp. A732).

could never be known without the construction of their concepts within pure intuition—again we let aside this point in the course of modern internalization. I thus also do not bother with the question of Kant's taking too much structure as *a priori* granted, e.g., the Euclidean space. Second, the proofs of propositions from these axioms can only mediate knowledge if they are also intuitively evident. The requirements for a real demonstration are twofold, it must show that the claim necessarily holds and it must be intuitively (self) evident:

> Nur ein apodiktischer Beweis, sofern er intuitiv ist, kann Demonstration heißen. [...] Aus Begriffen a priori (im diskursiven Erkenntnisse) kann aber niemals anschauende Gewißheit d.i. Evidenz entspringen, so sehr auch sonst das Urteil apodiktisch gewiß sein mag.[25]

Thus only mathematics could offer these demonstrations in the proper sense. For Kant, however, this essentially constructive approach is not restricted to geometry. Even the most 'mechanical' part of mathematics in Kant's horizon, namely algebra, works by using constructions, at least in the form of signs or characters. In fact, the process of proving—not only on the level of heuristics—is always accompanied by jotting down signs and pictures; every single step provokes our own constructions. There is no mathematics without the recognition of the basic signs. Again we also consider the macro-level: Intuition enables us to take a proof or argument as a whole—it is an ability of unification which could be illustrated by the difference between step-by-step calculation and an overall insight. This insight, however, is—as the search for virtue in *Menon*—hardly controllable. Only *after* a successful construction we might be able to give good reasons.

In my opinion it is sensible to keep *both* aspects, analytic and synthetic, as essential, but never completely realized ideals. Just from the outset, mathematical axioms, definitions and proofs are an *invitation* to follow the *free* mental constructions of the author; to repeat his or her synthesis. Though active construction is needed, though there is no proof without synthesis, you *might* always ask for a closer analysis which *enforces* the conclusion. And every proof communicates the *promise* that this further analysis could be done.

[25] "An apodictic proof can be called a demonstration, only in so far as it is intuitive. [...] Even from a-priori concepts, as employed in discursive knowledge, there can never arise intuitive certainty, that is, [demonstrative] evidence, however apodictically certain the judgment may otherwise be." Cf. (Kant, 1781, pp. B762).

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