Perspectives on mathematical practice from an educational point of view

Katja Lengnink[1] and Nikola Leufer[2,*]

[1] Fachbereich Mathematik, Universität Siegen, Emmy-Noether-Campus, Walter-Flex-Straße 3, 57068 Siegen, Germany

[2] Institut für Entwicklung und Erforschung des Mathematikunterrichts, Universität Dortmund, Vogelpothsweg 87, 44227 Dortmund, Germany

E-mail: lengnink@mathematik.uni-siegen.de; nleufer@math.uni-dortmund.de

1 Introduction

The mainstream research program in philosophy of mathematics is taking mathematical practice increasingly into consideration. Several publications during the last years serve as witnesses for this change (cf., e.g., Van Kerkhove, 2009; Mancosu, 2008; Heintz, 2000; Buldt et al., 2008)).

A practice-based philosophy of mathematics requires an understanding of what mathematical practice and mathematical knowledge actually are. This is not a trivial question with undisputed answers. Buldt et al. (2008) propose that discrepancies between philosophical interpretations should be approached by "supply[ing] more data from a more varied range of sources, including data established via accepted empirical methods" (p. 324).

We are encouraged by this proposal to provide some insight into the research field of mathematics education,[1] where studying mathematical practice—viz. the practice of elementary school and high school students students learning mathematics—is our everyday business. Mathematics education has established a variety of research techniques to pursue questions of educational interest. In this paper, we are providing an overview to philosophers of significant empirical data and our methods of interpretation.

However, mathematics education has more to offer to philosophers than just serving as a pool of raw material. The questions raised and results obtained by mathematics education are of relevance for philosophy of mathematics under the assumption that there is an analogy between scientific mathematical practice and the learning process of mathematics in school. One of the central claims of the introduction of (Bruner, 1977), the psychologist Bruner, a well-known representative of mathematics and mathematics education, is

*The authors acknowledge support of the *Wissenschaftliches Netzwerk* PhiMSAMP funded by the *Deutsche Forschungsgemeinschaft* (MU 1816/5-1).

[1]In the following, we use the expression "mathematics education" as a translation for the German term "Mathematikdidaktik".

Benedikt Löwe, Thomas Müller (*eds.*). *PhiMSAMP. Philosophy of Mathematics: Sociological Aspects and Mathematical Practice*. College Publications, London, 2010. Texts in Philosophy 11; pp. 209–233.

Received by the editors: 6 October 2009; 31 January 2010.
Accepted for publication: 14 February 2010.

> that intellectual activity anywhere is the same, whether at the frontier of knowledge or in a third-grade class-room. What a scientist does at his desk or in his laboratory, what a literary critic does in reading a poem, are of the same order as what anybody else does when he is engaged in like activities—if he is to achieve understanding. The difference is in degree, not in kind. (Bruner, 1977, p. 14).

This is certainly a strong assumption, but we would like to stress the fact that mathematical research activity is a way to achieve understanding which makes the analogy between learning mathematics and doing research in mathematics very plausible. Results in mathematics education, derived from the observation of learning processes, may therefore inspire and focus research on mathematics as a research discipline.

In Section 2 we shall give a rather short—and necessarily not comprehensive—overview on the aims and methods of research in the field of mathematics education. In Section 3 we sketch our work as reseachers in mathematics education on the basis of two surveys and give some results from our studies that have a direct connection with philosophical questions on mathematics. We shall conclude with a short discussion.

2 A didactical view on processes of learning mathematics. Research questions and methods

2.1 Questions and interests in mathematics education

What are researchers in mathematics education interested in and how do they approach their field of interest? A brief answer could be: Mathematics education focusses on the relationship between mathematics, humans and society (cf. Fischer and Malle, 1985). Thus, their research involves the background and the practice of teaching and learning mathematics throughout all age groups and all school forms. We work to obtain descriptive, normative, and constructive results. The term "descriptive" in our context refers to the description and analysis of learning and teaching processes. With the term "normative", we refer to questions like "What do we want to be taught and what do we want children to learn?" As we want our work to have immediate consequences for the mathematics classroom, we emphasize an additional constructive dimension; this means that results in mathematics education are translated into ideas and concepts that can be implemented in mathematics classrooms.

The German mathematical education society *Gesellschaft der Didaktik der Mathematik* (GDM) states the covered fields of interest and enquiry as follows (from their English-language webpage):

> Mathematical didactics focuses on the teaching and learning of mathematics within all age groups. It seeks to answer questions such as the following:

- What could, or should, students learn in mathematics education?
- How could, or should, mathematical contents be taught, how should mathematical ability be developed?
- What would enable students to derive pleasure from mathematical activity?

Mathematical didactics tries to find answers to such questions by

- Critical investigation with regards to justifying the contents and specific teaching aims within the framework of the general goals of mathematical education,
- Conducting research on learning conditions and teaching and learning processes related to the development of suitable empirical methods and theoretical concepts,
- Investigation and preparation of mathematical content, with the aim of making it accessible to specific learning groups.

Mathematics education tries to approach these questions by employing a range of results and research methods, also taken from other disciplines such as pedagogy, sociology, psychology, the humanities and history. Fields of interest in mathematics education can be mathematical content and the individual learning process. Mathematical content is investigated in a complexity of contexts, such as individual learning, the transmission process, the curriculum and general aims of education. On the other hand, the individual learning process of mathematics, the transmission process and other learning conditions and contexts themselves become the object of our study.

2.2 Research methods in mathematics education

Research interests and methods in mathematics education range from subject specific considerations, interpretative studies monitoring school situations to quantitative studies investigating comparative tests and general questions on the relationship of human, mathematics and society. A listing of research questions and approaches cannot be complete; we state some of them to give the philosophical reader an idea of the variety of issues; in our list, we emphasize the proximity to philosophical issues. In our account, we focus on research activities in Germany. Details and an international overview can be found in the "Second International Handbook of Research in Mathematics Education" (Bishop et al., 2003). Since in mathematics education, our concern is to understand mathematical thinking and mathematical procedures, most of our fields of enquiry are strongly dedicated to researching practice. For a philosophical perspective on mathematical practice, our results may be of interest when they touch the process of understanding mathematics between social conditions and human construction. In the rest of this section, we shall give examples of the following approaches:

- *Stoffdidaktik* as an approach of connecting basic concepts of mathematics with intuitions of the learners; (Example 1)
- Empirical investigations (qualitative and quantitative studies);
- Semiotic approaches;
- Sociological approaches; (Example 2)
- Beliefs concerning mathematics and mathematics education.

One field of enquiry in mathematics education genuinely linked to philosophical questions is called *Stoffdidaktik*. The main intention of *Stoffdidaktik* was to simplify and adapt mathematical topics for children. Since an increasing amount of material was to be handled, it became necessary to select and analyse appropriate content during the last decades. Some basic ideas and intuitions of mathematical concepts recur throughout several fields and levels of mathematics. These basic ideas and intuitions are constitutive for learning mathematics as they give some orientation and structure in a complex and diversified field of knowledge. *Stoffdidaktik* also deals with linking mathematics to general human thinking by investigating copulative concepts that allow for the transition between the real and the mathematical world. These basic concepts, regarded as mental models or mental objects are called *Grundvorstellungen* (Vom Hofe, 1995). Under underlying research questions will be approached by means of reflecting mathematics at all levels and researching children's beliefs and conceptions to mathematical objects, e.g., "What are the basic ideas in a certain field of mathematics, to what extent do they constitute it and how do they relate to general human thinking?" "Which conceptions and beliefs do learners have of mathematical objects and how do they relate to the basic ideas of mathematical concepts that are mathematically intended?" "Which educational value can be given to the mathematical object?"

On the one hand, *Stoffdidaktik* is very pragmatic as it opens out in the design of curricula and the construction of special learning arrangements. On the other hand, by aiming at the relation between human thinking in general and mathematical thinking, this very basic field of inquiry implicitly makes use of philosophical ideas and directly translates into philosophical issues.

Another branch in mathematics education examines classroom processes in various ways. There are, e.g., video based studies, differently designed and interpreted in many different ways following different methods. The well-known TIMSS Video Study is one of them (cf., e.g., Stigler and Hiebert, 1997). TIMSS shows and describes different forms of teaching and investigates the impacts on mathematical abilities of children.

For an analysis of the classroom videos, interpretative techniques are applied (cf., e.g., Heinze et al., 2006). By this means, hidden structures and differences in framing that affect the classroom situations from behind the scenes are brought forward; as, e.g., the actors' tacit convictions, experiences and expectations. It is interesting that what the participants perceive as mathematics turns out to be very different in these situations. Hence, mathematics can be looked at as a concept that does vary subjectively and therefore cannot exist independently but is closely bound to humans. If one also takes into account historical studies of how mathematical contents developed, this may generate an interesting question for the philosophy of mathematics: "If mathematics is not looked at as an independent body of knowledge, how is mathematics established in human thinking?" There are numerous studies involving videos showing processes of teaching and learning mathematics used that can shed some light on the philosophical questions.

Fundamentally different from the mentioned approaches are quantitative studies of intervention which are implemented to investigate the impact of certain training types or the use of certain media in classroom (e.g., specific training in problem solving strategies, use of computers, proof, ...). Intervention studies typically try to provide evidence for a growth of the level of competence in the group of learners that have undergone the specific training. It is not possible to name all the projects here. To give one example, quantitative projects have emerged from the DFG-funded program BiQua (e.g., Bruder et al., 2004, for problem solving).

Research in education also tries to measure competencies of learners at a given point in time. Large-scale quantitative surveys such as IGLU, PISA, TIMSS, VERA are well-known. These surveys use indicators for the assessment of skills and capacities and generate significant information about the abilities of groups. It is particularly noteworthy to observe how these studies understand mathematical competencies and to analyse the underlying conception of mathematics (cf., e.g., the concept of mathematical literacy underlying PISA in (Baumert et al., 2001), and the critical remarks in (Jahnke and Meyerhöfer, 2007)). On the other hand, specific evaluation may be necessary for some competencies to be actually perceived as mathematical skills. Hence, testing makes use of conceptions of mathematics and may also affect them. Investigating competencies considered valuable in teaching and research can provide background for philosophical discussions.

In particular the use of the computer as a tool suggests the question to what extent thinking and practice depend on the instruments used; is the use of different tools likely to alter our concept of mathematics? These questions are central in mathematics education; e.g., the semiotic approach by Dörfler (2008) and Hoffmann (2006), drawing on Charles Sanders Peirce, or the analysis of the usage of software in class and the resulting changes in practice and mathematics (e.g., Hölzl, 1994, on Cabri-Gometr-Software). The general philosophical issue underlying these studies is "What instruments are allowed in the construction and verification of knowledge?"

As Fischer and Malle (1985) point out, mathematics education draws on research within the complex web of relations between human, mathematics and society. Over the last years more and more sociologically oriented approaches in mathematics education (cf., e.g., Jablonka et al., 2001; Gellert and Hümmer, 2008; Leufer and Sertl, 2009) relate to this. Their work accounts for the learning individual by taking into consideration and reflecting his living and learning conditions. Mathematics education transgresses the borders of the discipline with such approaches that consider the practice of mathematics education as part of the subject of mathematics education; these are typically questions raised by the sociology of education and researchers in this field often employ sociological methods.

The ethnomathematical approach (references can be found in François and Van Kerkhove, 2010, in this volume) raises the issue of the relationship between human, society and mathematics from an interdisciplinary perspective, drawing on ethnology. Ethnomathematics is concerned with the social conditions of mathematical development and the development of mathematical competencies. This field of enquiry suggests a relativist perception of mathematics, looking and accounting for specific and different characteristics of mathematics in different cultures, groups or situations. Understanding mathematics as being socio-culturally shaped and characterised by the conditions of practice is very instructive for research on learning processes. Obviously, the ethnomathematical approach emphasizes the question of contingency, i.e., the question to what extent socio-cultural structures and settings influence the construction of knowledge, and whether different conditions could have or have resulted in a different development of the discipline.

Another field of enquiry in mathematics education that can be seen as a practical application of philosophical theories is the research on beliefs of students, teachers and parents (cf., e.g., Felbrich et al., 2008). Beliefs can be defined as subjective knowledge and theories and are closely related to the perception of mathematics and the ascription of relevance in everyday life situations. This relates to questions of educational value including issues of general education, claims of relevance in everyday life and also the establishment of mathematical literacy.

3 Mathematics education from an inside perspective. Two examples.

We shall expect the style of approach of mathematics education to mathematical practice or the practice of learning mathematics via the following two case studies. The first example is from the field of *Stoffdidaktik* and investigates the learners' conceptions of negative numerals and the historical and mathematical background. The second example illustrates a sociologically oriented issue and demonstrates the impacts of situative contexts on learners' competencies when working on mathematical problems.

Example 1: Basic concepts and student ideas—highlighting the relationship between humans and mathematics.

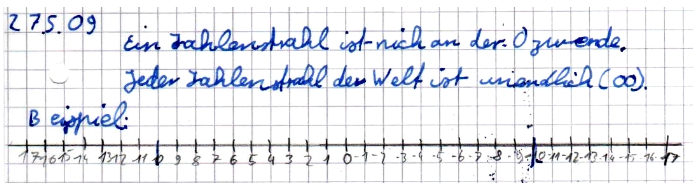
Looking at learning processes, researchers usually observe differences in the basic associations and concepts of learners and teachers. The intended mathematical understanding of theories and their concepts often is not self-evident. For example the calculation $2 - (-5) = 7$ is mostly calculated correctly, but appears mysterious to many children: The calculation rules for integers are used correctly, but from the viewpoint of the children the result does not make any sense. Subtraction in primary school is a process of reduction. Therefore, how can the result of a subtraction be more than the minuend? Such differences between the associations of the learners and the intended mathematical ideas and concepts are widespread and ingredients of every learning process. They give us pointers to the immanent obstacles of a mathematical subject, often the same that proved to be barriers in the historical development.

In the following example, we analyzed the ideas of 13 high-performing children in the fourth grade concerning negative numerals, focussing on the differences and semantic variances between intuitive and mathematical intended concepts. This investigation has been designed by Lengnink and Linnenbaum and implemented by Linnenbaum (2009). The students were asked to solve six different tasks and write down there solutions in a mathematical journal. The examples of the journal entries give an example of how the thinking processes of learners can contribute to clarifying the philosophical problems arising with the extension of the number range.

The semantic field of integers

On the basis of the question "What do you associate with negative numerals (*Minus-Zahlen*)?" the children were supposed to unfold their intuitive knowledge and their previous experiences with this mathematical subject. The children were asked to write into a mathematical journal. The results are widely spread in respect to different interests and outlooks.

Some children are interested in specific mathematical topics: Where do the negative numerals come from? When did they occur first? How do people calculate with them and why? Even the representation of integers was raised as an issue. Consider the diary entry of Ben who draws a number line and explains: “Ein Zahlenstrahl ist nich an der Null zuende. Jeder Zahlenstrahl der Welt ist unendlich (∞).”[2]



Negative numerals are often seen as the result of a subtraction task. Fabian writes “$100 - 1000 = -900$”, and then tries to calculate $500 - 634$. This process shows the difficulties and the uncertainty with the operation rules, as he writes: “$500 - 634 = -246$ glaub ich”.[3]

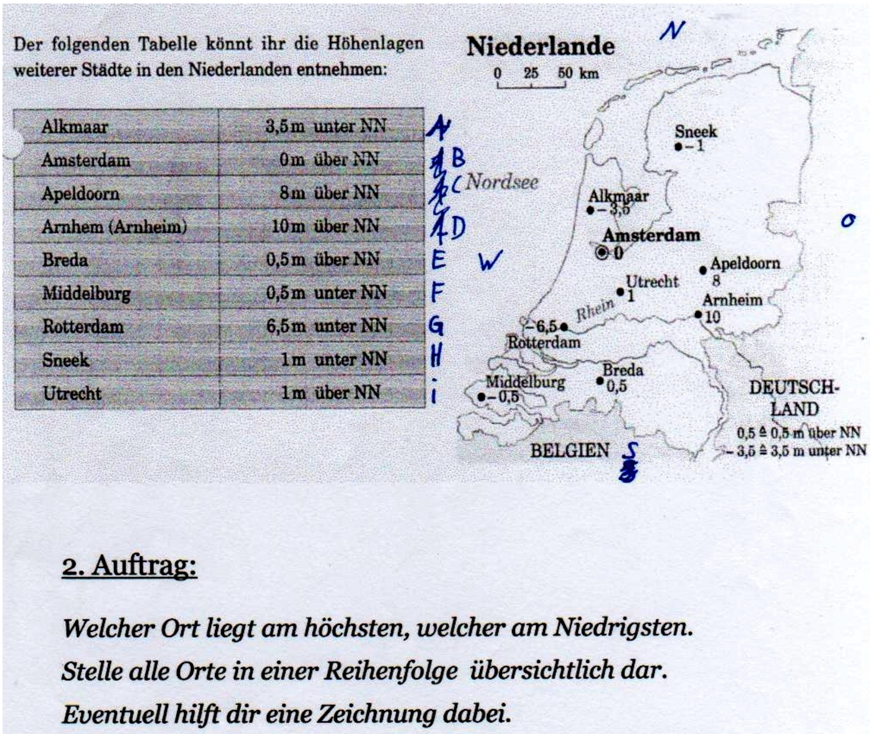
For some students it is important to search for counterparts from everyday life. Negative numerals are discovered on the thermometer or as debts. Some children mentioned the CD-player and the alarm clock as an example for negative numerals, where you can go back by a push button to get the last title or a former time (we are not entirely sure what the children were referring to).

In the children’s approach, a general phenomenon of development of number extensions arises. Whereas some learners accept the negative numerals as objects with certain calculation rules, others search for coherent contexts and sustainable associations for them in real life. They need such an analogue establishing a semantic foundation, which allows them to calculate appropriately.

The same phenomenon can be observed in the historical development. Whereas in China negative numerals occur as solutions of systems of linear equations (since 100 BCE), even Descartes did not want to accept them calling them “false solutions”. It is not just a reflection of the individual intellectual development of children if they find cognitive access to negative numbers difficult: historically, the concept of negative numbers needed time to be accepted.[4]

The intuitive associations and the mathematically intended basic concepts of integers very often do not match; in *Stoffdidaktik*, the *basic concepts* of negative numeral are extracted from a mathematical perspective in order to bridge this gap.

[2] A number line does not end in 0; every number line of the world is infinite (∞).
[3] $500 - 634 = -246$, I believe.
[4] Cf. (Seife, 2000; Alten et al., 2003).



Der folgenden Tabelle könnt ihr die Höhenlagen weiterer Städte in den Niederlanden entnehmen:

Alkmaar	3,5 m unter NN
Amsterdam	0 m über NN
Apeldoorn	8 m über NN
Arnhem (Arnheim)	10 m über NN
Breda	0,5 m über NN
Middelburg	0,5 m unter NN
Rotterdam	6,5 m unter NN
Sneek	1 m unter NN
Utrecht	1 m über NN

2. Auftrag:

Welcher Ort liegt am höchsten, welcher am Niedrigsten.
Stelle alle Orte in einer Reihenfolge übersichtlich dar.
Eventuell hilft dir eine Zeichnung dabei.

FIGURE 1. English translation: "You can find the altitudes of more cities in the Netherlands in the following table: Alkmaar 3.5 m below sea level; Amsterdam 0 m above sea level; Apeldoorn 8 m above sea level; Arnhem 10 m above sea level; Breda 0.5 m above sea level; Middelburg 0.5 m below sea level; Rotterdam 6.5 m below sea level; Sneek 1 m below sea level; Utrecht 1 m above sea level. Task 2: Which city is the highest; which is the lowest. Present all cities clearly in an order. Maybe a picture helps."

3.1 Mathematical basic concepts and student ideas

What do children do when they are asked to solve mathematical tasks with integers? The children were asked to work on the task listed in Figure 1. The students came up with several different representations. For instance, Ben drew a vertical section (Figure 2) and describes his figure as follows:

> ich habe mir die Aufgaben angeguck und sie nach dem Alphapet geordnett. So hatte ich es sehr leicht. Manche Striche gehen nach oben, manche gehen nach unten weil: es ein über und ein unter giebt[5]

[5] I looked at the tasks and ordered them [the cities] alphabetically. Therefore it was

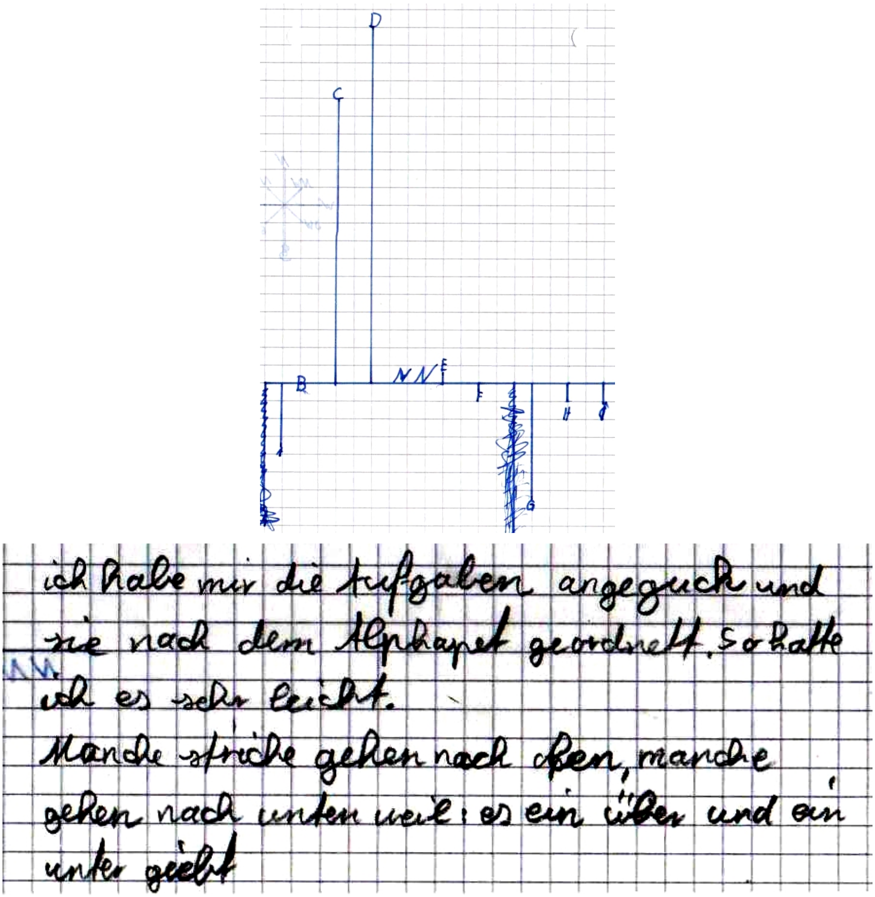


FIGURE 2.

With this representation he is able to solve all further tasks concerning altitude and temperature differences. Ben consistently interprets the subtraction as "up to".

This can be seen in Sven's solution as well who uses the analogy of height and temperature and represents the cities on a scale as given in Figure 3 (note that the zero is not written). Sven distinguishes between the sign of an integer and the arithmetic operator. The operator is marked in red and means "up to" and the sign is marked in blue and means "below sea level".

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very easy for me. Some lines go up, some go down, because there is an 'above' and a 'below'.

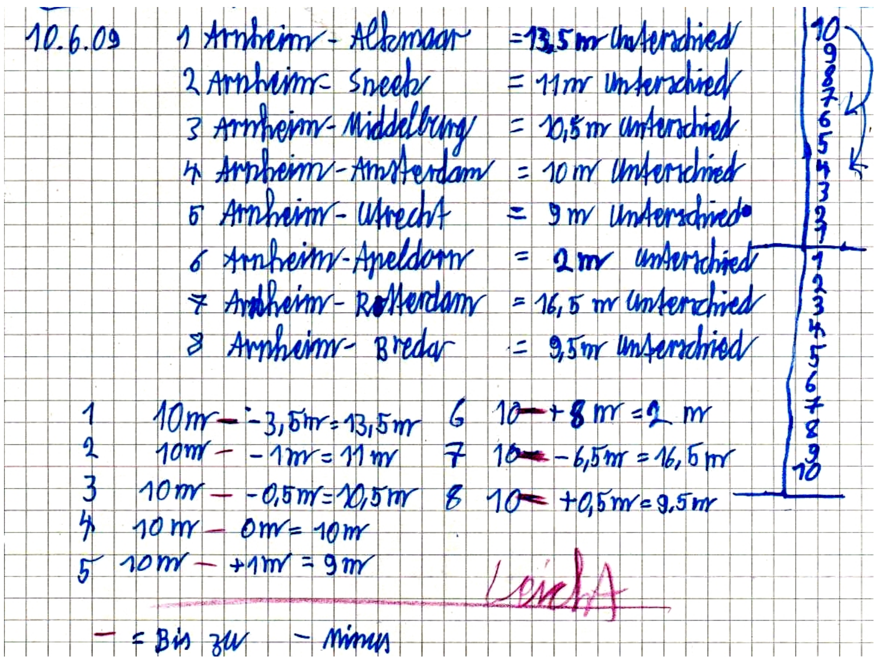


FIGURE 3.

This stresses two basic conceptions of integers which need to be developed in the learning process for an appropriate understanding: The description of a current state with integers (an actual temperature, height, . . .) has to be distinguished from a change of state (variation of temperature, variation of account, variation of water-level, . . .) which is also described by integers (cf. Postel, 2005). In the first interpretation − (minus) is a sign, in the second one we interpret − as an arithmetic operation. Sven seems to understand this difference intuitively.

It is an important step in the learning process to interpret negative numerals as natural numerals with a negative sign. This can also be detected as a step in the historical development of the integers (cf. Malle, 2007). As negative numerals occur in the process of solving systems of equations, it becomes necessary to find a notation and calculation rules for those numerals. This was a desideratum at one particular time in the historical development of mathematics and similarly, this is exactly what Fabians asks for: he wants calculation rules. Even though he calculates correctly, he comments (Figure 4: "ich weiss nicht wie mann über und unter zahlen

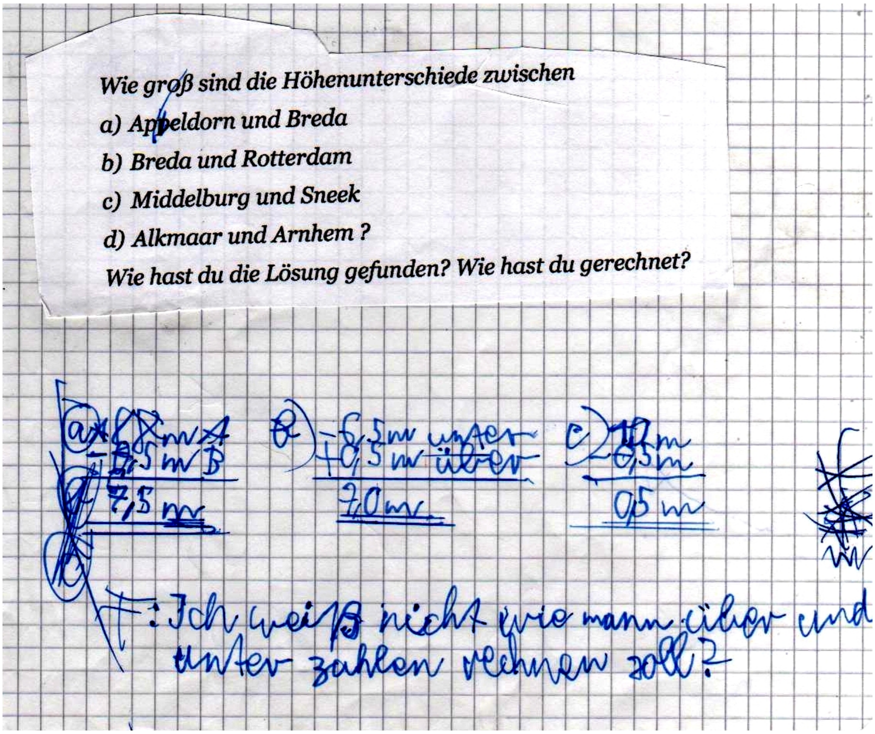


FIGURE 4.

rechnen soll".[6] He probably has problems dealing with the non-standard notation of his calculations. This problem is not shared by others. For example, Nadine confidently describes her handling of integers (Figure 5: "mann rechnet die umkeraufgabe und das ergebnist die − Zahl".[7] Nevertheless there are also children who have difficulties calculating with and interpreting integers. The order especially causes difficulties. For instance, Nils puts the cities above and below sea level in different categories (Figure 6) and comments: "Ich habe erstmal die Uberstehende Orte von 0 m–10 m, dann habe ich die unterstehende Orte von 1.5 m–6.5 m geordnet."[8] The numbers are interpreted as positive integers in both groups. Even after introducing the number line as a representation tool for integers, there is still room for interpretation. Being on the left-hand side is not automatically

[6] I do not know how to calculate with "above" and "below" numbers.

[7] One calculates the inverse task and the result is − numeral.

[8] I first ordered the "above" cities from 0 m to 10 m, and then the "below" cities from 1.5 m to 6.5 m.

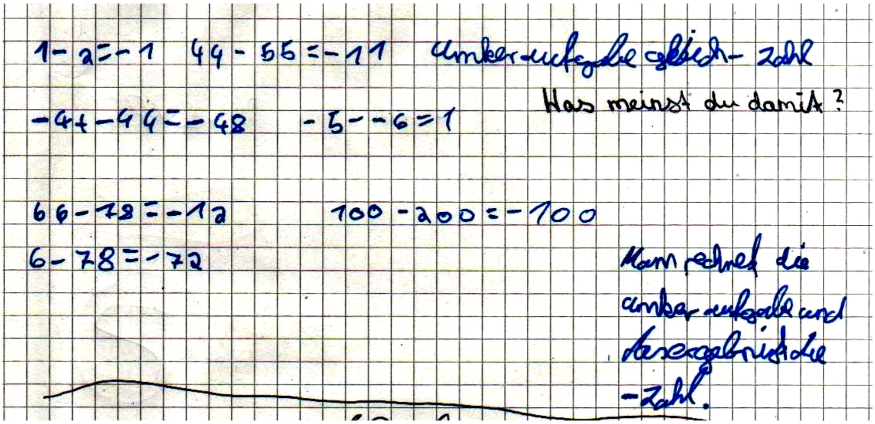


FIGURE 5.

interpreted as being smaller. For instance, Nils argues: “Für mich ist −5 größer als −3, denn es liegt weiter von der 0 entfernt.”[9]

Considering the context of temperature and debts the mathematical order sometimes does not make any sense at all. The encountered tensions and frictions can be clearly seen in Jonas’s journal (Figure 7). Even though his order of integers matches with the mathematical one, he is confused:

> Das stimmt nicht. Denn −2 C° ist höher als −11 C° Unter Null sind es mehr das stimmt aber ich finde trotzdem −2 C° ist höher. Das hatte ich ja auch schon in den Aufgaben dafor begründet. Wenn man schon Schulden hat und dann trotzdem noch mehr abhebt wird er größer. Wenn man beim Kontostand noch mer abhebt wird er kleiner weil er dann noch tiefer sinkt.[10]

Finally, the tension between everyday life and mathematics can be seen in the following comment as well (Figure 8): “Anja rechnet nach der Große der Zahlen. Bea rechnet mathematisch.”[11] Postel (2005) points out that the usual interpretation of the order of integers has to be discussed in school lessons. The interpretation “is less than” has to be combined with the concept of the number line and the left-right-orientation.

[9]For me, −5 is bigger than −3 since it has greater distance from 0.

[10]This is not right since −2 C° is higher than −11 C°. It is more below zero, that’s right, but I still think that −2 C° is higher. I had already argued for this in the last tasks. If you are in debt and withdraw even more money, then it becomes bigger. If you withdraw more from the account balance, it becomes smaller as it falls even more.

[11]Anja calculates with the size of numerals. Bea calculates mathematically.

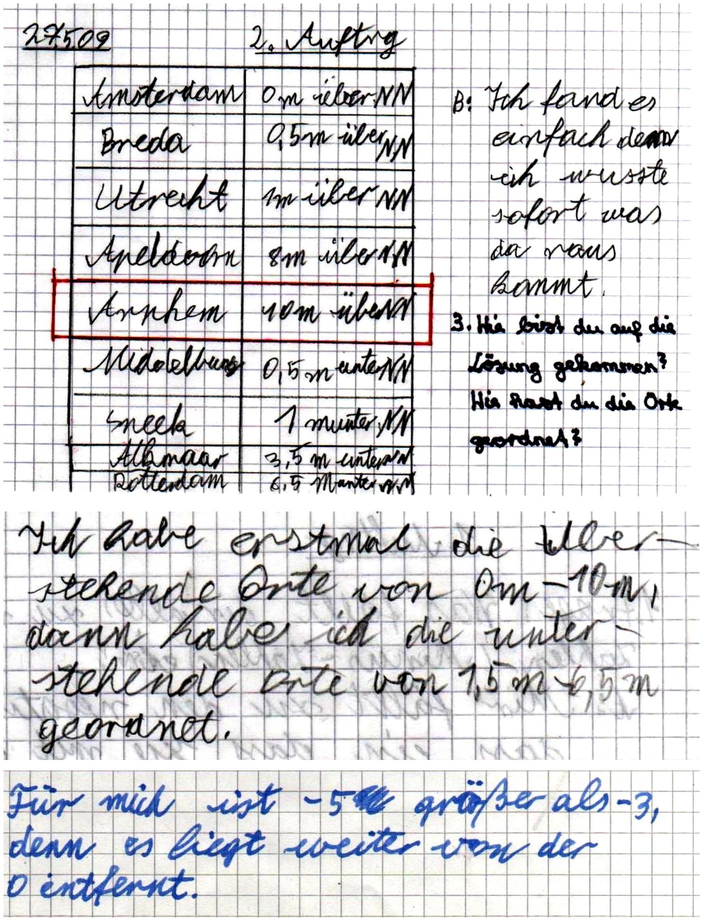


FIGURE 6.

Conclusion Example 1

The problems of learners in the field of integers can be clarified by an analysis of the historical development and the intuitive student ideas. They are based on the fact that the basic concepts in this mathematical subject are very abstract. While it was possible for the number ranges introduced prior to the integers (natural and positive rational numerals) to illustrate the concept by way of concrete experiences and handling with concrete objects, this cannot be done here. Vom Hofe (1995) therefore distinguishes between primary and secondary basic concepts; the basic concepts occurring in the field of integers are secondary concepts. They have to be built

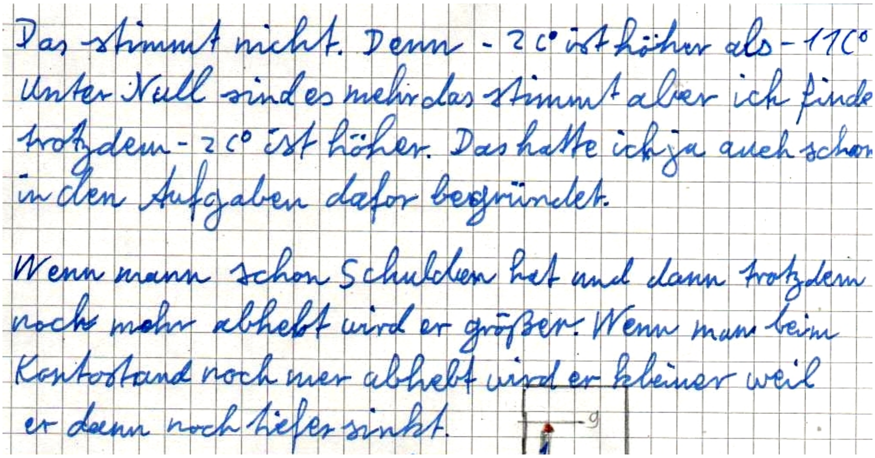
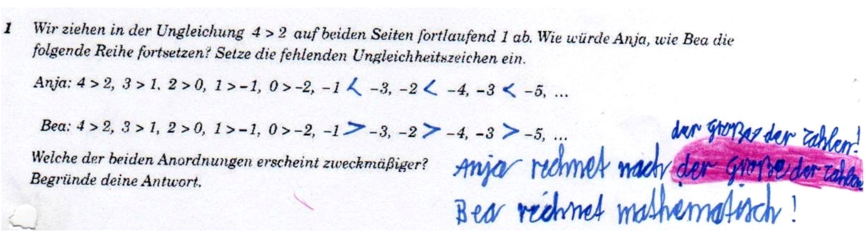


FIGURE 7.



1 *Wir ziehen in der Ungleichung 4 > 2 auf beiden Seiten fortlaufend 1 ab. Wie würde Anja, wie Bea die folgende Reihe fortsetzen? Setze die fehlenden Ungleichheitszeichen ein.*

Anja: 4 > 2, 3 > 1, 2 > 0, 1 > −1, 0 > −2, −1 < *−3, −2* < *−4, −3* < *−5, ...*

Bea: 4 > 2, 3 > 1, 2 > 0, 1 > −1, 0 > −2, −1 > *−3, −2* > *−4, −3* > *−5, ...*

Welche der beiden Anordnungen erscheint zweckmäßiger? Begründe deine Antwort.

FIGURE 8.

up by experiments with abstract representations such as the number lines, because a direct correspondent for integers does not exist in everyday life (cf. Postel, 2005, p. 196). The objects corresponding to negative numerals are not concrete and physical, they have a theoretical status. Taking this into consideration, the learning obstacles become obvious.

Links to philosophy

In the learning process, natural links between general human thinking and mathematical thinking appear, although there are also differences and semantic displacements between the intended mathematical concepts and self-evident ideas. The presented analysis exemplifies how learner perspectives open up our view of the status of the objects in a certain fields of mathematics. Furthermore, the learners' confusions can clarify the discrepancy between intuitive thinking and mathematical formalization. They highlight the implicitness of semantic shifts in the understanding of numerals—

historically and in the personal development. In the example, this can be illustrated by the order of integers, which is counterintuitive and is in conflict with the order to cardinal extent. This issue was an obstacle in history as well and it needed some time to overcome it with an elegant conceptual design.

The semantic content of the integer symbols could be of philosophical interest as well. How is the meaning of integers established? Which semantic shifts occurred in history and how were they triggered? How did integers get accepted? Malle (2007) points out that usually in both the learning process and history there is often formal calculation without any interpretation, purely mathematical. Only later on do explanations and meaning emerge, as acceptance often comes along with familiarization and figural representation of mathematical objects. In the above example, the acceptance of the integers was reached together with the axiomatization of numerals: This progress may be ascribed to Ohm. His claim was not to define terms and numerals by their properties, but to look at their properties in operations and relationships. Emphasizing operational properties instead of some complex inherent nature has considerably eased, and hence promoted, the handling of integers in historical development (cf. Alten et al., 2003, p. 305). Today there are only few children who do not believe in the existence of negative numerals, but for many children they are still mysterious (cf. Malle, 2007, p. 53).

This sort of reflection about the relationship of humans and mathematics rests on the assumption that the growth of knowledge as well as the obstacles to it are analogous in personal and historical development (as indicated in the given example above). To find out which concepts that we have accepted by long-term exposure, it may be interesting to investigate learner perspectives on mathematical subjects, in order to get an exposure-free view of the mathematical concept itself. This methodology, the so-called didactical reconstruction, has stood the test in the didactics of natural sciences (cf., e.g., Duit et al., 2005). From this point of view, mathematics is not independent from the learners. Devlin points that out as well: music is not music if it is only written down on a sheet of paper. It has to be played and listened to, to become music; and the same holds for mathematics. Mathematical symbols aren't mathematics, they have to be thought about by living persons (Devlin, 2000).

3.2 Example 2: Mathematics and everyday knowledge: What's the right discourse?

For many reasons, including the mediocre German PISA results, there is increasing demand for a stronger orientation towards the application of mathematical concepts in the mathematics classroom. The OECD has coined

Aufgabe: „Schätzen" 2

Die Klasse 10a möchte einen Ausflug zum Düsseldorfer Stadttheater machen und dort eine Vorstellung besuchen. In der Klassenkasse sind noch 550 Euro. Reicht das Geld aus der Klassenkasse, um den Ausflug zu bezahlen?

FIGURE 9.

the popular notion of "mathematical literacy" as the individual's capacity to "[...] use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen" (Baumert et al., 2001). Consequently, efforts have been made to fill mathematical syntax with content to emphasize a real-life relevance and prevent mathematics from degenerating to solely technical operations. The aim is to achieve a meaningful handling of mathematical symbols and concepts so that children will develop the competence to apply mathematics in a more confident and flexible way.

As so-called "realistic items" embed a mathematical problem in the context of a (more or less) authentic problem situation, this type of problem is of great importance in most of today's mathematics classrooms. Realistic items connect the real and the mathematical world as shown in the example[12] given in Figure 9. The task, entitled *Schätzen* (estimation) is about a tenth grade class that wants to undertake a field trip in a nearby city to go to the theatre. Ticket price and class size are not given. The question is: "There are 550 Euros left in the class treasury. Is this enough to pay for the trip?" This problem was part of a set of realistic problems given to tenth grade students of a secondary school. The whole interview session with each student took about one hour and took place in the school building. The study's initial interest was to analyse, from a mathematics didactical perspective, how students handled the realistic contexts and how fast and to what extent they would draw on their everyday knowledge to solve the problems. For the interpretation of the students' work, the theoretical model of pedagogic practice by the sociologist Basil Bernstein (e.g., Bernstein, 1996) was used.

In this paper, we shall use the example to focus on the relation between mathematics and real world argumentation and raise the issue of

[12] The item is taken from a workshop presentation, held by Andreas Büchter, Dortmund 2006.

legitimate reasoning. As becomes clear by adopting a sociocultural perspective throughout the example, the student's struggling with the mathematical problem turns out to be a—subjective—conflict of determining allowed methods. While in school this issue may be often settled by explicit teachers' expectations and a strong classroom culture that is—presumably—not so in mathematical research. The question of acceptable methods therefore remains an interesting challenge for studying mathematical practice.

Mathematical capabilities and realistic items

Prija is a quiet immigrant girl with respectable marks in mathematics. This is how Prija worked on the problem:

1	S	[reads the problem out loud]
2		But we do not know how much this costs [looks at interviewer] [7 sec silence]
3		It only says that it is . . . er. . . there are 550 Euro in the class treasury, but how much the theatre is, per person, it does not say.
4	I	Hmm
5	S	[appears to read the title] Ok, maybe one is to estimate, how much the tickets are [8 sec pause]
6	I	Go ahead
7	S	Er . . . but we do not know the number of people. . . .
8	S	[aborts working on the problems]

What can we say about the student's mathematical competencies by studying the transcript? First, Prija most probably recognizes the underlying mathematical model of the problem. That is presumably why she needs the quantities "class size" and "ticket price", which must be multiplied to calculate the entire costs. Supposedly, Prija would then match the result with the money available and would solve the problem by this means. From this perspective, she probably does have the technical mathematical abilities to solve the problem, but fails to deal with the problem of the missing values.

Those missing values need to be estimated—as already specified in the title of the item. This means the student needs to come up autonomously with strong assumptions from her everyday experience. These assumptions are strong or rather crucial in the sense that it will follow from this data whether the answer to the problem is "yes" or "no".

Negotiating uncertainties, as required in this task, can be seen as official practice in today's mathematics classrooms. Handling crude data and estimation is, after all, a part and consequence of realistic problem solving and is therefore discussed, practised and also examined in the mathematics class. But, when seeing estimation as a mathematical competence or mathematical ability, then Prija does not have the competencies necessary to solve the problem.

Mathematics in a social context

It is now interesting to take a closer look: What is it exactly that the student does not know or is not capable of? Prija definitely has problems with estimation. This becomes very clear when she reads the title of the problem "Estimation"—which in effect tells her exactly what she is supposed to do. But, although the interviewer confirms her idea, she does not put the hint into practice. What do these difficulties mean? After all, it is very improbable that a tenth grade student with respectable marks in mathematics cannot answer the question "Estimate how many students are there in a (your) tenth grade class?" It seems more likely that Prija does not understand what exactly she is supposed to do. And, because of her confusion, she stops working.

But, if the student is able to estimate values as required in the problem, but does not make use of this in this situation, then how can we describe the competencies that she lacks?

To do so, we have to look more thoroughly at what exactly is to do at that very point and what might be the cause that hinders the student to demonstrate her abilities. We shall now try to account for the whole setting: The student is used to a very traditional mathematics class and maybe therefore perceives it as strongly classified, which means strongly insulated from other disciplines. In this case study she finds herself in an exertive interview situation that she associates with mathematics class. And she has to work on problems that she may consider as strongly classified as well. From experience or by habit she expects this problem to be stated in an explicit way, requiring a precise and unique answer. To solve the problem however, she has to recognize that the item "Estimation" does not refer to this strong classification, but foils it. The linkage to an authentic realistic situation requires a new practice which, to the student, may not seem legitimate: She has to refer to her everyday life and bring in crucial information from her everyday knowledge. This means the item draws on the student's domestic context, her social background—that potentially differs so radically from the official school context that she is accustomed to ignore and repress it during her time at school. The required ability for the above item at this point is to successfully recognize the context and negotiate confidently between home and domestic or unofficial and official skills and contexts. As this does not match with what the student considers as legitimate reasoning, she stops working on the problem.

The analysis of the transcripts obviously ends up describing a competency that neither is part of the school curriculum nor is evenly spread among the population. Using the terminology of Basil Bernstein, we can reformulate the interpretation: Given the interview situation that Prija may perceive as strongly classified, the student runs the risk of not recognizing the weak classification of the problem. She does not select the "recognition rule" that allows her to recognize the situation and the required discourse. As Bernstein and others have found out, this is mainly a characteristic of children from the working class (e.g., Bernstein, 1996; Cooper and Dunne, 2000). Their findings include that differences in socialisation, particularly in language use (such as familial child communication) seem to have a strong impact on the distribution of recognition rules.

We can conclude from these considerations that Prija might have solved the problem under different conditions that would have helped her to recognize the weak classification of the problem. Different conditions could either mean a different interview framework or a slightly different wording of the task, pointing out the weak classification more explicitly. Other children from the same class but from different social backgrounds might have handled the problem more successfully.

Conclusion Example 2

Videographing, transcribing and interpreting working processes using the transcript turns out to be a very reasonable method to obtain in-depth ideas and results from an interview situation. The consistent wish to achieve understanding of students' thoughts, workings and problems with this method can obtain interesting results and questions of educational interest. This may be a useful method to investigate in mathematical practice as well.

However, to explain, analyse or even predict Prija's problem, we have to abandon the level of the immediate situation and take more parameters into account: This is where Basil Bernstein's model of pedagogic processes can be of help: The model offers a language, which is capable of linking the microlevel of immediate interaction, the institutional level and the institutional macrolevel of society. The sociological perspective, drawing on Bernstein, systematically enables us to consider as many parameters.

When we stop focussing exclusively on the mathematical problem and instead take into account the active learner, his situative contexts and his societal involvements, boundaries between professional and non-professional, technical and non-technical, official and non-official competencies start to blur. Especially for researching realistic items, which is one example where school and domestic contexts and discourses explicitly and implicitly overlap, a powerful sociological perspective is beneficial (cf. also Gellert and Jablonka, 2009).

Adopting Bernstein's perspective and accounting for as many different levels in the above example, the legitimacy of mathematical knowledge not only becomes an issue, but an issue of social importance. This means, it is not only in question, to what extent it may be legitimate to employ real-life arguments in mathematics. But we can also ask: Who might use to what extent and why real life arguments—and who does not?

As a conclusion from the above considerations, we seem to get at least two results that may be of interdisciplinary interest: On the one hand, the example raises the question to what extent real-life argumentations are acceptable in mathematics classrooms. Generally spoken, this questions the disciplinary legitimacy of arguments and points to Fleck's notion of "thought-styles" in "thought-collectives" (cf. Fleck, 1980).

On the other hand, looking at mathematical practice thoroughly involves some uncertainty as to what we are actually observing and what contexts we are actually dealing with. The examples firmly suggest a situative perspective and an understanding of mathematics as always closely bound to persons—learners or scientists—and situations.

In mathematics education, such a perception of mathematics helps and requires to appreciate not only precise solutions but to account for the subjective constructions of our students. The mathematician and philosopher of mathematics Reuben Hersh (1997) offers a socio-constructive concept of mathematics that seems very suitable for our job: He consistently understands mathematics as human activity (for details, cf. also Prediger, 2004): "[...] mathematics must be understood as a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context: I call this viewpoint 'humanist'."

As shown by empirical research, mathematics turns out to be not clear-cut at all, but subjective and of some social relativity. Understanding mathematics in a "humanist" way may be instructive and helpful not only for mathematics education. It will surely be an insightful prerequisite for studying mathematical practice as well.

Summary

In this paper we have attempted to give a rough idea of the variety of research questions and approaches of (German) mathematics education. Our concern was to explain the specific interests of mathematics education to the non-specialist and to promote (some of) our approaches. We have outlined in two research examples how the interest in the process of handling mathematics and in the context in which mathematics is practiced leads to a deeper understanding of mathematical objects and mathematical practice itself.

The mentioned studies emphasize the proximity of didactical and philosophical questions and research in the field of mathematical practice. Maybe the philosophy of mathematics can use some methods and questions of mathematics education as a blueprint to investigate the practice of mathematicians. Assuming the analogy of learning and researching mathematics (cf. Bruner, 1977) we expect helpful results for our field of understanding learning processes better.

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