Embodied strategies in mathematical cognition

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1 Introduction

Most traditional theories of cognition, such as the computational theory of mind favored by cognitive science during the last half of the 20th century, imagine cognitive content to be located inside the head of the thinking agent. The head is, in short, conceived as a container isolating the cognitive content inside from the physical world on the outside. Furthermore, sensing, planning, and acting are supposed to be three clearly distinct activities, and they are supposed to be performed in the mentioned order: First you sense, then you use your internal cognitive resources to form a plan, and finally you enact your plan.

This general picture of thinking is also visible in some theories of mathematical cognition, such as the 'abstract code model' where a tripartition between comprehension (of the mathematical problem), calculation, and response (e.g., in the form of a written number) is hypothesized (Campbell and Epp, 2005).

During the 1980s this conception of cognition was met with considerably opposition from many different fronts (such as studies in animal vision, robot engineering, philosophy of consciousness, neuroscience *etc.*). I will not review all of the arguments here, but instead focus on a single line of criticism. This line of criticism simply points out that the container metaphor is inadequate. Parts of human cognition can indeed be described as contemplation taking place inside the head, as the container metaphor suggests, but not all of it. Much of human cognition can only be understood as interactive processes involving the brain, the body, and the surrounding environment (both social and physical).

The interactive nature of human cognition is evidenced by our use of a number of cognitive tools including:

1. Epistemic actions, i.e., actions taken in order to gain knowledge and not in order to achieve a practical end.

2. Cognitive artifacts, i.e., artifacts developed in order to facilitate thinking. These can be either physical or conceptual.

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3. Conceptual metaphors, i.e., metaphors where a—typically abstract—phenomenon or idea is expressed and understood using the terms and knowledge from another—typically more basic—conceptual domain.

It should be noted however, that these tools are not neutral instruments that merely enhance our cognitive powers. The tools might shape our cognitive style and they are in some instances constitutive of particular types of reasoning.

In what follows, I will briefly present the cognitive tools and exemplify their use in mathematics. I will demonstrate how the three tools complement each other in the mathematical practice, and discuss to what extent they have shaped what mathematics is.

2 Epistemic actions and distributed cognition

Our first example of interactive cognitive processes that involves brain, body and environment is *epistemic actions*. They were identified and defined by Kirsh and Maglio (1994). Where pragmatic actions are actions performed with a pragmatic goal, such as peeling potatoes, an epistemic action is defined as:

> "[A] physical action whose primary function is to improve cognition by:
>
> 1. Reducing the memory involved in mental computation, i.e., space complexity;
> 2. Reducing the number of steps involved in mental computation, i.e., time complexity;
> 3. Reducing the probability of error of mental computation, i.e., unreliability." (Kirsh and Maglio, 1994, p. 514)

Kirsh and Maglio give several examples of epistemic actions. If you have a tendency to forget your key, you might for instance leave it in your shoe. Then you are sure to be reminded of it when you put on your shoes before you leave your apartment. By putting the key in the shoe, the key becomes a cognitive device that reduces both the probability of error and the demands on internal memory. Hence, the act of putting your key in your shoe is an epistemic action.

The concept of epistemic actions fits well in the general theory of cognition called *distributed cognition*. In contrast to traditional theories of cognition, this approach does not identify cognition with information processing going on inside the human skull, but allows cognition to be distributed across individual human minds, social groups, and resources in the external environment (Zhang, 1997; Holland et al., 2000).

In the example with the key given above a cognitive task was solved by exploiting already existing environmental resources. Distributed cognition however, also includes another class of cases where cognitive tasks are solved by the use of specially created *cognitive artifacts*.

The midwife's wheel is a good example of such an artifact. The wheel consists of two discs, one marked with the months of the year and the other with the weeks 1 to 42 of a normal pregnancy. Given the date of conception, the discs are aligned in accordance with a simple algorithm, and both the due day and the current duration of the pregnancy can be read off from the artifact. This artifact allows the user to substitute complicated mental calculations with epistemic actions; she simply manipulates the disks and reads off the result. Notice, that the artifact in this example does not enhance or strengthen cognition. Instead, mental calculations are substituted with an entirely different task: the correct alignment of discs.

Artifacts might also change or influence the total outcome of a task. To use an example most academics might have experienced, a talk delivered by heart is different form a talk read from a manuscript. And a talk given by the help of Power Point is something different again. Paper and Power Point-presentations serve as excellent external substitutes for memory, but they do more than that. The two artifacts offer different possibilities, they make different things easy and hard, and that might change not only the style of the talk, but also the content. In other words, cognitive artifacts are not always neutral tools that simply make it easier for us to do, what we have always done. They might in a more profound way change us and influence the nature and content of the cognitive tasks we perform.

3 Cognitive artifacts in mathematics

It is not hard to find examples of cognitive artifacts used in mathematics. The very basic operation of counting is commonly supported by external scaffoldings in the form of tally marks, pebbles, fingers or other objects at hand. Multiplication and other arithmetic operations can be supported by a number of different devices such as written tables, counting boards, abaci, computers, and calculation machines. All of these devices allow mental calculations to be substituted by different types of epistemic actions; using written tables, calculations are substituted by perceptual and search processes, using abaci and counting boards, calculations are substituted by manipulation of physical counters in accordance with given algorithms, and using computers, calculations are substituted by keyboard operations.

Written tables have been in use at least since the second millennium BCE (where tables of multiplication, reciprocals, square- and cube roots were used by the Babylonians (Kline, 1990, pp. 5)). The origin of the abacus and counting boards are more uncertain. The oldest known counting board is the Salamis tablet from the fourth century BCE (Menninger, 1992, p. 299), but the invention of such devices is probably much older; according to linguistic evidence, primitive tallying boards were already used by the Sumerian civilization in the fourth millennium BCE (Nissen et al., 1993, p. 134). So it seems as if the use of physical artifacts and the embodied strategy of substituting mental calculations with epistemic actions is an integral part of mathematical cognition and has been more or less so from the dawn of calculation.

The widespread use of such artifacts raise a number of interesting philosophical questions, not least the question of impact, i.e., has the introduction of such artifacts altered the practice and content of mathematics?

This question is especially pressing in current years as the use of one type of physical artifact—electronic computers—might alter the mathematical practice in a number of ways with the introduction of computer assisted proofs, experimental mathematics, and new visualization techniques. The mathematical community has not yet decided exactly what stand to take on these issues, but it seems likely that an acceptance of one or more of these new techniques will have an impact on both the epistemic standards and the content of mathematics (Tymoczko, 1979).

3.1

Apart from physical calculating devices such as those mentioned above, the most important artifacts in modern mathematics are written symbols. In order to shed some light on the cognitive significance of the use of symbols, I will begin with an in-depth cognitive analysis of the Hindu-Arabic numerals. The Hindu-Arabic numerals are no doubt one of the most successful systems of mathematical notation, and by gaining a better understanding of the cognitive role played by the symbols used in the system we might get some understanding of the role played by mathematical symbolism in general.

In one such analysis, Zhang and Norman (1995) compare the performance of Hindu-Arabic, Greek alphabetic, and Egyptian hieroglyphic numerals on multiplication tasks. Zhang and Norman conclude that the superiority of the Hindu-Arabic numerals can (at least in part) be explained by the fact that, compared to the other systems, they allow most of the steps of the multiplication algorithm to be externalized; i.e., performed as epistemic actions using pen, paper, and the numerals of the system in question.

Unfortunately, this analysis suffers from several weaknesses. Most importantly, all three systems of numerals are compared on the same polynomial algorithm for multiplication.[1] I find this questionable, as it is unlikely

[1] In a numeral system with base x, a number a can be represented in polynomial form as $\sum a_i x^i$. In this representation the algebraic structure of polynomial multiplication of two numbers a and b is: $a \cdot b = \sum a_i x^i \cdot \sum b_j x^j = \sum \sum a_i x^i b_j x^j = \sum \sum a_i b_j x^{i+j}$

that either the Greek or the Egyptians actually used this algorithm. We do not know much about how the Greeks did their calculations, but they probably performed the actual calculation on abaci or counting boards, and only used the alphabetic numerals as a way to record the result (Menninger, 1992, pp. 299). For this reason, the proper unit of cognitive analysis in this case is the numerals and the counting board in combination.

The Egyptians on the other hand used a binary algorithm and not a polynomial algorithm for multiplication.[2] Using the binary algorithm with Egyptian hieroglyphic numerals, multiplication can quite easily—and to a large extend externally—be performed as a series of doublings and reductions. Thus, looking only at the ratio between internal and external workload cannot tell us why or if it is easier to use the Hindu-Arabic than the Egyptian hieroglyphic numerals. Other factors, such as the total number of operations performed, must be considered as well.

In order to get a better understanding of the unique qualities of the Hindu-Arabic numerals, we might instead use the typology of numeral systems developed by Stephen Chrisomalis (2004, cf. our Table 1).

In this typology, the Hindu-Arabic numerals are characterized as positional and ciphered. We shall analyze these two characteristics one by one starting with ciphering.

In comparison with cumulative systems, the advantage of ciphering is the possibility of a much more compact way of writing numbers. The number eight for instance can be written with a single symbol in the Hindu-Arabic system: '8', whereas it takes eight symbols to represent the same number in the Egyptian hieroglyphic system: 'IIIIIIII', and four symbols using Roman numerals: 'VIII'.

Due to the compactness of the script, one would expect calculations in general to take fewer operations in a ciphered than in a cumulative system. Unfortunately very little empirical work has been done in this area, but the hypotheses is backed up by at least one study (Schlimm and Neth, 2008), where the ciphered Hindu-Arabic system is compared with the cumulative Roman system. Using virtual agents to perform a large number of addition and multiplication tasks in ways similar to human agents, Schlimm and Neth found that the number of basic operations, such as perceptual steps, attention shifts, and motor actions, was considerably more numerous when using Roman numerals than when using Hindu-Arabic numerals.

The compactness of ciphered numerals however, comes at a price. In a cumulative system, the value of each power of the base is represented by a repetition of a specific symbol. So for instance, in Egyptian hieroglyphic

[2] Cf. (Katz, 1998, pp. 8); in the binary algorithm, the multiplier a is decomposed into its binary representation $\sum a_i 2^i$ and the multiplicand b is multiplied with each term: $a \cdot b = \left(\sum a_i 2^i\right) \cdot b = \sum a_i 2^i b$.

	Additive (sign-value) The sum of the value of all the numerals gives the total value of the whole number	**Positional** The position of each numeral decides which power of the base the numeral is to be multiplied with.
Cumulative Many signs per power of the base. These are added to obtain the total value of that power.	Egyptian hieroglyphic Roman	Babylonian sexagesimal cuneiform
Ciphered Only one sign per power of the base. This sign alone represents the total value of that power.	Greek alphabetic	Hindu-Arabic
Multiplicative Two components per power, unit-sign(s) and power-signs. These are multiplied to give the total value of the power.	Chinese traditional	LOGICALLY EXCLUDED

TABLE 1. Typology of numerical notation systems (redrawn with small adjustments from Chrisomalis, 2004, p. 42).

system eight tokens of the symbol 'I' means eight, and eight tokens of the symbol '$\bigcap$' means eight tens (i.e., eighty) and so forth. In other words, in a cumulative system there is an iconic likeness between the value of a power and the number of signs used to represent this value. This is not so in ciphered systems. The sign '8' gives no clue to the fact that its value is eight. In a ciphered system the numerals are conventional symbols, and their values must be remembered. Or differently put: The numerals are meaningless symbols until interpreted.

In sum, from a cognitive point of view the choice of a ciphered over a cumulative system is in fact a trade-off, where a reduction in the number of operations is obtained by increased demands for internal cognitive work (cf. also Schlimm and Neth, 2008).

Turning to the positional character of the Hindu-Arabic system Zhang and Norman (1995) might be right in pointing out that a positional system allows for an easy separation of the power and base dimensions; the power of each numeral of a number is represented by its position, and the base value by its shape. Due to this fact, calculations can easily be broken down into two simpler tasks: 1) calculations involving only the numerals 0 through 9 and 2) the recording of the result of such calculations in the right positions on the paper.

In ciphered, additive systems such as the Greek alphabetic both base and power values are represented by the shape of the individual numeral. In the Greek system eighty is represented by a single symbol 'π', and the reader will have to infer both the base value—eight—and the power value—tens—from the shape of the sign. Calculating using such a sign, you will either have to separate the base and the power dimension in order to reduce the calculation to simpler facts, or you will simply have to memorize the necessary tables for all the numerals used in the system. As such systems need unique numerals for each unit of each decimal order, the tables get very big, and consequently memorizing them pose a considerable challenge to long-term memory.[3] Separating the dimensions on the other hand greatly increases the demands on internal, mental work-load (cf. Zhang and Norman, 1995, for a detailed analysis of the last). Either way, the written numerals of ciphered, additive systems does not seem to offer much support for calculations.

In conclusion, the Hindu-Arabic numerals are a very special kind of symbols. Unlike the symbols of ciphered, multiplicative systems, the Hindu-Arabic numerals allow calculations to be performed largely externally as series of epistemic actions, and unlike the numerals of cumulative systems, they are conventional, i.e., abstract symbols that have no iconic likeness with that, which they represent. Due to these characteristics, the Hindu-Arabic numerals serve as a very effective cognitive tool. They allow calculations to be (largely) externalized and performed as purely formal manipulations of the symbols; you do not need to worry about the meaning of the symbol as you calculate, you only need to remember the correct algorithms and transformation tables, and then the symbols will take care of the rest. The numerals, in other words, allow you to perform calculations as epistemic actions in ways similar to those allowed by physical artifacts such as the abacus or the midwife's wheel. Only, when you use the numerals you have to write down the transformations of the stings of symbols as you go along instead of simply manipulating preexisting physical tokens.

This cognitive characteristic of the Hindu-Arabic numerals is particularly interesting, as the very same characteristic applies to the symbols used in modern mathematics. The symbols are (mostly) conventional signs that have no likeness to that, which they represent. This allows computations to be performed as series of epistemic actions, where the symbols are treated as physical objects and manipulated according to strict, formal rules until

[3]The Greek system for instance has 27 different numerals, resulting in a multiplication table with 729 entries

a result can simply be read off from the paper. The modern symbols are a cognitive artifact, allowing mathematical computation to be externalized as a formal and meaningless game with uninterpreted symbols functioning as physical tokens (De Cruz, 2005).

The power and enormous influence of this cognitive tool is witnessed by the fact, that the formalist movement simply identified mathematics with "manipulation of signs according to rules" (Hilbert, 1926, p. 381). From my point of view, this is a mistake. The use of external cognitive artifacts is simply a tool used to do mathematics, and one should not take the tools for the trade. Although this particular cognitive strategy is at the heart of modern mathematics, it does not exhaust what mathematics is. As we shall see in the next section, other cognitive strategies are equally important.

3.2

With a few exceptions (such as Diophantus's notation for powers Kline, 1990, pp. 138), abstract symbols apart from numerals were only introduced to mathematics in the late 16th and (mainly) 17th century by mathematicians such as François Viète, René Descartes, William Oughtred and John Wallis.

The introduction of this new cognitive artifact has had a very clear impact on both the epistemic standards and the content of mathematics.

As an example of the first, I will mention the use of formalizability as a criterion on the acceptability of a proof. Although proofs are rarely given as rigorous formal deductions, most mathematicians will only accept a proof if it is somehow made probable that it could be formalized and given as a series of purely formal transformations in a formal theory (cf. Tymoczko, 1979, pp. 60). Such a criterion quite clearly only makes sense in a praxis involving abstract symbols. Furthermore, it rules out many of the proofs created before the introduction of abstract symbols, especially all proofs relying on diagrams in a non-trivial way. In sum it seems clear, that the use of abstract symbols have had an enormous impact on the epistemic standards of mathematics, i.e., the standards used to judge whether a knowledge claim is acceptable or not.

As an example of an impact on content, I will turn to geometry, where the idea of substituting calculations with manipulation of abstract symbols primarily was carried out within the paradigm of analytic geometry, founded by Descartes and Pierre de Fermat in the 17th century. This new paradigm quickly led to new and very powerful ways to solve problems which were difficult or even impossible to solve by traditional synthetic means. Take the advent of the calculus as an example; here the extremely difficult problems of constructing tangents and determining areas were replaced by (more or less) mechanical syntactic transformations of symbols. However, the paradigm of analytic geometry also led to a change in the conception of which objects were accepted in geometry. In analytic geometry, any curve which can be given an equation is accepted. This is clearly an expansion in comparison to the traditional Euclidian framework, where only objects which can be constructed using straight lines and circles were admitted. Ultimately, the adoption of analytic methods led to the discovery of several curves (such as space filling curves and continuous, but nowhere differentiable curves) which were strictly impossible from a geometrical point of view. However, because these objects could be expressed analytically they were accepted anyway (and the geometrical intuition was rejected). So all in all, the introduction of symbols as a cognitive artifact has had a clear impact on the content of mathematics, i.e., which objects to accept.

A number of other types of impact could also be discussed. De Cruz (2005) suggests, for instance, that abstract symbols might serve as a way to 'anchor' semantically opaque concepts (such as square roots of negative numbers) in concrete external representations. This idea is highly interesting, but it is unfortunately slightly problematic. Abstract symbols are only one of many ways to represent mathematical content. Square roots of negative numbers can for instance be represented symbolically (as "i"), rhetorically (as "the square root of minus one") or even diagrammatically using the complex plane. When it comes to computations, abstract symbols are clearly superior to rhetoric forms of representation because symbols as we saw above allow computations to be externalized and performed as epistemic actions. However, it is not clear that symbols are similarly privileged when it comes to anchoring opaque content. In fact, complex numbers were at first discovered and handled using purely rhetoric means (in Gerolamo Cardano's *Ars Magna* from 1545). So, De Cruz might be right in noting that abstract symbols are in fact used to anchor opaque content, but it is not clear that they are necessary means of doing so. Perhaps content could be anchored just as well by purely rhetoric or other means.

4 Conceptual metaphor

The use of epistemic actions and physical cognitive artifacts implies that human cognition is embodied in a very concrete sense. A disembodied mind cannot put a key in her shoe or operate a tool such as the midwife's wheel. Our physical body offers possibilities and has limitations for interacting with the surrounding world, and these constrains on our bodily interactions condition which artifacts we can and cannot use.

But that is not all. Our body and basic bodily experiences also influence our cognitive life in a much more profound way. As it turns out, we seem to use basic life-world experiences as a way to structure abstract thinking. This structuring is revealed by our heavy use of metaphors taking basic life-world experiences as their source-domain in our conceptualization of abstract phenomena (Lakoff and Johnson, 1980).

Examples of such metaphors are easily found in everyday language. Take for instance the expressions: "I couldn't quite grasp what he was saying", "Everything he said just flew over my head", and "Did you get it?" In all of these examples ideas are conceptualized as physical objects, understanding is conceptualized as grasping or holding such objects, and an exchange of ideas is understood as an exchange of objects. Consequently, the situation of attending a lecture can be understood as a situation, where the lecturer is throwing objects to the audience. When objects are thrown from one person to another, it might be difficult for the receiver to catch the objects, the thrower should be careful to aim his throw at the receiver and so on. In other words, something as abstract as learning and understanding can be structured and understood using the basic bodily experience of someone throwing things at us.

As pointed out in Lakoff and Johnson (1980, pp. 46) ideas can be conceptualized using a wide range of other metaphors. Ideas can be seen as living organisms; they can be born, mature, get old and die, and they can come to fruition or be planted in someone's mind. Ideas can be understood as food; they can be hard to digest, half-baked, rotten, fresh, or hard to swallow. Or ideas can be seen as cutting instruments or weapons: They can be sharp, dull or cut right to the heart of matters.

All of these metaphors help us understanding and structuring the abstract phenomena of ideas using well-known and concrete everyday experiences. The various metaphors highlight different aspects of the target-domain and offer guidance in different situations; When giving a lecture, you should be careful to aim what you are saying at the audience in order for them to catch your ideas, and when going to a debate (which is commonly conceptualized in terms of warfare!) it is wise to bring ideas at least as sharp as those of your opponent.

The cognitive approach to metaphor was introduced in the late 1970's, most influentially by Lakoff and Johnson (1980). According to this approach, the structuring of a concept in terms of concrete experience is not something exceptional or rare. In fact: "[...] metaphor is pervasive in everyday life, not just in language but in thought and action. Our ordinary conceptual system, in terms of which we both think and act, is fundamentally metaphorical in nature" (Lakoff and Johnson, 1980, p. 3).

A point of debate is the exact cognitive significance of such metaphors. When we talk about ideas as objects using the metaphorical expression "I couldn't quite grasp what he was saying", do we also *think* of ideas as objects, or is the metaphor merely a linguistic phenomenon? It is not very hard to find examples of 'dead metaphors', i.e., metaphors which might once have had cognitive significance, but clearly do not have so any more. Take for instance the expression: “I have examined 14 students today”. The word ‘examine’ originates in the Latin word *examen*, which literally means ‘tongue of a balance’. So, examining student is—or was—originally a metaphor, where the process of judging the knowledge of a student was described by comparing it to the process of weighing goods at the marketplace. Today however, most English speakers do not know the original meaning of the word ‘examine’, and they do not think of balances or processes of weighing when they use it. The word has simply obtained a new meaning, and consequently the original metaphor is dead and has ceased doing any cognitive work.

So, how do we know that not all of the metaphors discussed above are dead? This is a very important question that needs to be answered before the cognitive approach to metaphor and language can carry any philosophical weight.

The cognitive approach argues their case by putting forth different types of evidence. One type of evidence is studies in linguistics. Most people would probably not accept expressions such as: “That idea is hard to swallow, I fear it won’t grow at all”. The two parts of the sentence express the same phenomena of disliking an idea, but still, the sentence seems to be somehow inconsistent. The reason for this inconsistency is the fact that the sentence contains two metaphors exploiting two different source-domains; edibles and living organisms respectively. And although ideas can be understood as both edibles and organisms, they cannot be done so at the same time. The metaphors used to conceptualize ideas must in other words be applied in a coherent way. This suggests, that the analogies to basic experiences expressed in the metaphors still have cognitive significance and structure not only the way we talk about ideas, but also the way we think about ideas.

Similarly, a sentence such as “that idea is full of vitamins” will probably be understood immediately and effortlessly by most people familiar with English, even if it is the very first time they hear the expression. This suggests that the analogy exploited in the ideas-are-food-metaphor is still active and allows us to use knowledge of nutritional facts to understand new aspects of the abstract domain of ideas.

Apart from such linguistic evidence, the basic claim of the cognitive approach is backed up by empirical evidence from neural science. A full review of this evidence is unfortunately beyond the scope of this paper, so I will restrict myself to a more thorough review of the evidence put forth in connection to the metaphors of mathematics (below). The reader is referred to Lakoff and Johnson (1999, pp. 36) for a comprehensive list of the different types of evidence.

5 Metaphors of mathematics

Imagine teaching school children the fact that a negative number multiplied by a negative number gives a positive number. The children might simply learn the rule: this is how it is, this is how we play the game of arithmetic. But simply learning the rule does not give the children any comprehension of why it might be so.

If the children are to understand why a negative number multiplied with a negative number gives a positive number, you might start telling them about the number line and explain multiplication in terms of movement and locations in space. It might go something like this: Negative numbers are opposite numbers. Where positive numbers are located to the right of 0, negative numbers are located at similar distances, but to the left of 0. Multiplying any number b with a positive number a simply means walking the distance from 0 to a, and then keep on walking until this distance has been repeated a total of b times. Then the number located at the point where you end up is the result of the multiplication. As negative numbers are opposite numbers, multiplying any number a with a negative number b, you will have to walk exactly the same distance as before, only in the opposite direction. So, if a is a negative number located to the left of 0, multiplying it with another negative number b, you will end up walking to the right, and consequently you end up at a location inhabited by a positive number. Hence, a negative multiplied by a negative gives a positive.

This story might explain why negative numbers multiplied by negative numbers give positive numbers, but none of it is literal. Numbers are not locations on a line, and multiplication is not movement. The explanation making the formal operations meaningful is made up of metaphors, or, to be more precise, it is an inference from one basic metaphor where arithmetic is conceptualized as motion along a path and numbers as locations on the path.

As pointed out by George Lakoff and Rafael Núñez, this metaphor is only one out of three basic metaphors used to conceptualize arithmetic (Lakoff and Núñez, 2000, pp. 54). In the other two, numbers are respectively conceptualized as collections of objects and as objects constructed by other objects. These conceptualizations are visible in expressions such as: "If you add three and four, you get seven," "If you put two and two together, you get four," "Five is made up of two and three," and "If you take three from seven, how much do you have left?" (examples from Lakoff and Núñez, 2000, pp. 54). Lakoff and Núñez also add a fourth metaphor, the measuring stick metaphor, where numbers are conceived as locations on a measuring stick. Based on the linguistic evidence, however, it is hard to distinguish this metaphor from the more general metaphor, where numbers are conceived as locations on a line or path.

Lakoff and Núñez claim the basic laws of arithmetic, such as commutativity, associativity, and even closure of the natural numbers to be derived from the structure of the source-domains of these metaphors. To take one example, we know from basic experience that adding a collection of objects to another collection of objects always results in a collection of objects. When numbers are metaphorically conceptualized as collections of objects, we can infer that the addition of two numbers must always result in a number, and consequently $\mathbb{N}$ must be closed under addition (Lakoff and Núñez, 2000, p. 60).

The exact status of this claim is unfortunately somewhat unclear. Is it a genealogical claim, i.e., a claim about how the laws of arithmetic were once established, or is it a psychological claim about how each individual person understands mathematics? In either case more evidence is needed in order to support it.

The use of metaphors in mathematics is not restricted to basic arithmetics. Metaphors are frequently used in more advanced subjects as well. Calculus for instance, draws heavily on metaphors of physical movement. Although all central concepts such as function, continuity, differentiability etc. are now defined in terms of sets, i.e., discrete and motionless entities, in textbooks functions still oscillate, approach, tend to etc. (cf. Núñez, 2004, for an interesting treatment of several examples from textbooks).

In the examples given so far all metaphors take life-world experiences as their source-domain, but metaphors taking already accepted theories as their source-domain also play an important part in both the expansion and the unification of mathematics. As an example of the first, William R. Hamilton's use of mappings between geometry and algebra in his discovery of the quarternions deserves mentioning (Pickering, 1995, cf.), for an interesting analysis of this case). As an example of the second—unification—the modern reduction of virtually all mathematical entities to sets is an obvious case. Numbers are not literarily, only metaphorically sets, lines are not literarily infinite sets of points, and so on.

Turning to the question of the cognitive significance of the metaphors, we shall continue the discussion opened above by reviewing some of the empirical evidence used to argue, that the metaphors of mathematics have real cognitive significance.

One line of evidence comes from the study of gestures. In short, there seems to be a close link between gestures and speech, and it is believed that gestures and speech both reflect the same cognitive processes. A study of college professors teaching calculus suggests that the metaphors of movement used by the professors are active, i.e., when the teachers use words of movement they also think in terms of movement (Núñez, 2004, pp. 68). In a revealing example, a professor describes how the values in an infinite sequence oscillate between two fixed values. While he gives this description, he waves his hand from side to side and presses his thumb and index fingers together as if he was holding a small object. The professor seems to be thinking about the two bounding values as fixed locations in space, and about the values in the sequence as a tiny object which is moving—literally oscillating—between the two bounding locations.

The study of gestures gives strong support to the thesis, that the metaphors used by the professors actually reflect how they think—at least in that particular situation. But, the teaching situation is a very special situation, and quite often teachers use colorful metaphors for purely didactical reasons; the abstract subject must be related to something well-known by the students. This however, does not give us any proof that the professors still think in terms of movement when they return to their offices and start doing mathematical research. The study of gestures cannot give us the final answer to the fundamental question: To whom and when are the metaphors active?

Another line of evidence comes from experimental psychology and the study of the human brain. A number of experiments suggest that at least the natural numbers seems to be encoded in the form of some sort of magnitude. Reaction time experiments for instance, reveal that the time it takes a subject to judge whether a number represented in Hindu-Arabic digits is larger or smaller than a given target depends on how close the given number is to the target; the lesser the distance between target and given number, the longer the reaction time (Dehaene et al., 1990). Furthermore, studies of patients who have lost part of their mathematical capabilities due to injuries of the brain suggest a close connection between basic arithmetic skills, body maps, and spatial maps (Dehaene, 1997, pp. 189). This has been used to support the view that at least basic arithmetic is closely connected to basic life-world experiences of the body and physical space (Lakoff and Núñez, 2000, pp. 23).

This type of evidence should be treated with much care. The evidence is still somewhat inconclusive, and in addition it is questionable to what extend studies of the physical brain can tell us anything about mental phenomena such as thinking and understanding. Does the (supposed) fact that arithmetic is encoded in the same region of the brain as spatial- and body maps really prove, that we understand arithmetic using bodily and spatial experience? Even granted such an intimate relation between the physical brain and understanding, the evidence only supports a limited connection between life-world experiences and mathematics. Only arithmetic is (apparently) located in the brain area in question, while other mathematical capacities are not. This was, for instance, the case in a patient suffering from acalculia (Hittmair-Delazer et al., 1995). Due to the effects of cancer treatment the patient had lost the ability to solve even basic arithmetic problems such as $2 + 3$ and $3 \cdot 4$. Nevertheless, he was still able to solve abstract algebraic problems such as recognizing that $(d/c)+a$ is not in general equal to $(d+a)/(c+a)$. Does this mean that arithmetic is closely connected to life-world experiences, but that algebra is not? And does it prove that all metaphors connecting algebra to space, the body, or physical objects are dead, while metaphors of arithmetic are always alive?

The thesis that the metaphors of mathematics are always active and determines the way mathematicians think, seems to me to be much too strong to be defended. Furthermore, it focuses solely on one cognitive tool. The above review of the cognitive tools used by mathematicians suggest another more modest hypothesis to me: perhaps the cognitive tools, I have reviewed, compliments each other. Mathematics can be done as a formal game, where external symbols are treated and manipulated as meaningless tokens of a cognitive artifact, or mathematics can be done as something meaningful, where the meaning—at least in part—comes from cognitive metaphors adding flesh and blood to the naked symbols.

As we have seen, the Hindu-Arabic numerals are conventional, abstract symbols that have no resemblance of collections or any other objects the numbers might signify. This allows for easy calculations: you do not need to understand anything. As long as you know the formal rules, pen, paper, and the written symbols will take care of the rest. But on the other hand, the symbols and formal rules can be treated as something meaningful, when we metaphorically conceive of numbers as collections, as locations in space and so on. Similarly, calculus is formalized in set theory, which allows for powerful formal calculations, but still, college professors use metaphors of motion when they teach calculus in order to ground the students' understanding of the abstract mathematical theory in bodily experiences.

The reader should keep in mind, that the above is only a description of the cognitive behavior of mathematicians and students. The normative aspect, i.e., how one ought to behave, falls outside of the scope of this paper. The description of the cognitive behavior inspires at least one normative question: Mathematicians seem to use metaphors, but should they keep doing that or was it better if mathematicians practiced mathematics as a meaningless, formal game?

6 Is mathematics special?

Finally, we might return to the overall theme of the conference PhiMSAMP-3: is mathematics special? This is a very general question which could be understood in many different ways. In this paper, we have seen that mathematics is not special from an epistemic point of view. Mathematical knowledge is obtained using the very same cognitive tools as many other kinds of human knowledge.

This might lead us to wonder whether mathematical knowledge consists of necessary and absolute truths. One might argue that it does not matter how we get to know, what we know: All mathematical truths are absolute and universal, no matter how they are obtained. Such an absolutistic stance seems hard to defend. As I have demonstrated above, cognitive tools are not neutral. The cognitive tools we use are essentially anthropomorphic; our body and environment determine both which basic experiences that are available to us as source-domain for conceptual metaphors, and which physical artifacts it is possible for us to produce and operate. Human knowledge is in a non-trivial way embodied, and thus shaped by the nature and possibilities of our body and physical surroundings. And that goes for mathematical knowledge as well. Consequently mathematics cannot be absolute and universal—at least not in any strong sense of those terms.

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