Loss of vision: How mathematics turned blind while it learned to see more clearly

Bernd Buldt[1], Dirk Schlimm[2,*]

[1] Department of Philosophy, Indiana University–Purdue University Fort Wayne (IPFW), 2101 East Coliseum Boulevard, Fort Wayne, IN 46805, United States of America

[2] Department of Philosophy, McGill University, 855 Sherbrooke Street West, Montreal QC, H3A 2T7, Canada

E-mail: `buldtb@ipfw.edu`; `dirk.schlimm@mcgill.ca`

1 Introduction

1.1 Overview

The aim of this paper is to provide a framework for the discussion of mathematical ontology that is rooted in actual mathematical practice, i.e., the way in which mathematicians have introduced and dealt with mathematical objects. Using this framework, some general trends in the development of mathematics, in particular the transition to modern abstract mathematics, are formulated and discussed. Our paper consists of four parts: First, we begin with a critical discussion of the notion of *Aristotelian abstraction* that underlies a popular folk ontology and folk semantics of mathematics; second, we present a conceptual framework based on the distinction between *bottom-up* and *top-down* approaches to the introduction of mathematical objects; in the third part we briefly discuss a number of historical episodes in terms of this framework, illustrating a general move towards top-down approaches and resulting in changes of the nature of mathematical objects; finally, the effects of this change with regard to the role of visualization in mathematics are discussed.

That mathematical objects are *abstract* posed a significant problem for philosophers already in ancient Greece. However, it is a commonplace that in the 19th century mathematics became *more* abstract.[1] What this 'more' consists in, we claim, can be explicated as a shift from a traditional notion of abstraction that goes back to Aristotle to a non-Aristotelian conception of abstraction.[2] This is closely related to the trend we identify in the development of 19th century mathematics, which reveals an increased attention to the study of mathematical *relations* as opposed to mathematical objects,

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[1]See, e.g., (Ferreirós and Gray, 2006; Gray, 2008).

[2]For a similar point, regarding theories in psychology, see (Lewin, 1931).

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based on the increased emphasis of top-down characterizations as opposed to bottom-up ones (see Section 3). We argue that these developments are best understood from a *structuralist* perspective, as opposed to a traditional Aristotelian view that is based on the notion of *substance*.

1.2 Structures

To our mind, the development of mathematics during the 19th century shows, certainly not in each single move mathematicians of the time took, but in their overwhelming majority nonetheless, a clear tendency to prepare and level the ground for 20th century structuralism. The term 'structuralism,' however, is ambiguous and can mean at least two different things.[3]

There is, first, the structuralism of the Bourbaki group, inspired by advances in set theory. Here one starts with a set of elements, a domain D, and then defines by set-theoretic operations alone a number of relations R over D that obey certain axioms, thus yielding a structure $\mathfrak{S}$ composed of D and R.[4] As such it can be seen as a basically bottom-up approach: one starts out with intuitively given objects conceived of as elements over which first-order quantifiers can range (like the natural numbers or points in a plane) and then defines a relational super-structure by set-theoretic constructions (identifying relations with certain n-tuples of the Cartesian product, etc.). Studying the properties of a structure in this context entails knowing how it is built up, its *Bauplan*. This reading of Bourbakian structuralism is well-known in particular among philosophers as it resembles the approach taken by model-theoretic semantics.

There is, second, the structuralism of those who champion category theory, inspired by advances in algebra.[5] Here one starts with a class of structured objects $A, B, C, \ldots$ (complex objects that are already equipped with structural features), a class of mappings $f, g, h, \ldots$ among them, such that those mappings have a number of desirable or 'natural' properties, and then studies those transformations that preserve the structure of those objects. As such it is a top-down approach: one starts with objects that usually are quite complex (like all sets, all topological spaces) and assumes them as already given. Studying the properties of a structure in this context entails knowing what their structure-preserving transformations are or what structures admit such transformations, but doesn't require us to know the structure's *Bauplan*.[6]

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[3] For a more detailed discussion of different versions of structuralism, see (Reck and Price, 2000).

[4] See, e.g., (Bourbaki, 1950, §3, esp. p. 225 ff.).

[5] See, e.g., (Awodey, 1996).

[6] And to the extent that a certain structure is characterizable in the language of category theory—say, a group $(G, \cdot)$ as a category with a single object $\star$ and all elements a of G as morphisms $a : \star \to \star$, or a poset as a category in which there is at most one

While in general we lean towards an interpretation of structuralism as inspired by category-theory, the notion of structuralism underlying this paper is more broadly defined and also more vague. For lack of an established term and for reasons to be adduced below we shall call it ‘non-Aristotelian structuralism.’ We shall be concerned with what many felt (and many still feel) is a development towards a mathematics that is more (sometimes too) ‘abstract’ and much less intuitive than it should be—a mathematics that features objects not very amenable to visualizations; many think this a sufficient reason to dismiss these developments. We think what causes this uneasiness is a conflict, not well-understood and hence unresolved, between, on one hand, a ‘folk ontology and semantics’ that starts with concretely given objects and their properties and, on the other hand, a non-Aristotelian structuralism that does not need such objects.

2 Folk ontology and folk semantics

We are surrounded in our daily lives by middle-sized concrete objects that have properties conveyed to us through our senses; this, we are inclined to think, captures what the furniture of the world is. This was also the starting point for Plato; but things quickly proved to be much more difficult.[7] When he set out to refute the Sophists and in particular their claim that there is neither truth nor falsehood but that man is the measure of all things, he was faced with two opposing viewpoints that had emerged from Ionian natural philosophy. There was, first, Heraclitus’ doctrine that the true nature of things—which love to disguise themselves and trick us into holding mere subjective opinions[8]—is to be in constant flux propelled by never ending opposition. Second, there was Parmenides’ doctrine that what truly exists is eternal and immutable, implying that the language of change is deceptive and that everything we can hope to know must therefore be statements that hold without any exceptions. These were serious issues of the time. Plato’s teacher Cratylos inferred that, based on Heraclitean doctrines, it is impossible for a language to have a denotational semantics and decided to stop speaking but to point with his finger instead, while Antisthenes, a student of Socrates like Plato, but later following Parmenides’ lead, found it only possible to argue for the truth of analytical sentences and limited his utterances to sentences like ‘a man is a man.’ Plato needed to develop

arrow between any two objects—then this doesn’t reveal the ‘familiar’ *Bauplan* of the structure either.

[7]Scholars disagree on what the correct interpretation of Plato in the context of his time is. We cannot hope to settle any of these disputes here; all we can do is to clearly say where we stand, i.e., to acknowledge that our own account of Plato’s philosophy of language is heavily indebted to Rehn (1982).

[8]‘Nature loves to hide itself’ (φύσις κρύπτεσθαι φιλεῖ) is one of his more famous statements; see (Diels and Kranz, 1952, frg. **B** 123) or (Marcovich, 2001), frg. 8.

a theory of language that was able to refute the relativism of the Sophists and to establish that declarative sentences can indeed be true or false; at the same time he needed to accommodate Heraclitean and Parmenidean arguments and find a way to reconcile the idea of permanent objects, assumed to be one, with fleeting properties, which are many (the venerable problem of 'unity *vs* multiplicity'). Plato's solution to this entangled knot of problems was to develop a comprehensive theory of language and then base crucial arguments upon that theory. Due to his efforts—and for his times this was quite an accomplishment—Plato might have very well been the first to clearly identify the grammatical structure of subject and predicate as underlying declarative sentences; a sentence according to Plato always is an artful 'composition' (σύνθεσις) or close 'intertwining' (συμπλοκή) of 'nouns and verbs' (ὀνομάτων καί ρημάτων).[9]

Aristotle adopted Plato's basic insights about the linguistic functions of nouns and verbs but not his teacher's conclusions (e.g., that knowledge of the physical world—knowledge here understood in its emphatic, Parmenidean meaning of the word—is not possible as the world forms a realm of change, becoming, not of being). Both, however, agree with Parmenides that any sentence is 'about something' (περί τινος), where this 'something' always refers to a 'state of affairs' (πρᾶγμα).[10] For Plato, this was just a necessary condition to ensure the 'matter-of-factness' that characterizes any declarative sentence, while Aristotle extended the 'about something' structure to a 'something about something' structure (τι κατά τινος). This, then, was according to Aristotle the proper structural analysis: A declarative sentence (λόγος ἀποφαντικός) features a subject S (ὑποκείμενον) and a predicate P (κατηγορία), where the 'predicate something' is about the 'subject something,' or, as Aristotle would also formulate it, P is 'predicated of' (κατηγορεῖν) or 'belongs to' (ὑπάρχειν) S.

Unlike Plato, who addressed the Sophistic and Ionian challenges mainly in the realm of language, Aristotle took an ontological turn—the solution, he remarked, belongs to another field of investigation[11]—and stipulated that the grammatical subject S always denotes (σημαίνει) an 'ousia' (οὐσία),[12] and it was this concept of ousia (or, the 'what is' (τί εστι) that was meant to shoulder the main bulk of explanatory work.[13]

[9]See Plato, *Cratylus* 424e–425a; *Sophistes* 261c–262d.

[10]See Plato, *Sophistes* 263a, resp. *ibid.* 262e; Aristotle, *Topica* I.8, 103b7; English translation can be found in (Cooper, 1997; Barnes, 1984).

[11]Aristotle, *De Interpretatione* V, 17a15.

[12]Defined as what can be predicated but is never predicated of; see Aristotle, *Categoriae* V.

[13]Like in the case of Plato, there is quite some disagreement among scholars on the details of a proper understanding of Aristotle's ontological doctrines, his '*prima philosophia*' (πρώτη φιλοσοφία). As we are only interested in the mainstream views that emerged from it, we feel free to gloss over all these difficulties. We skip in particular Aristotle's quite

We can capture the basics of what we need in the following by modifying the account given by Spade (1985, p. 236ff). An individual substance s, denoted by a grammatical subject S, does not change and thus allows for knowledge, but also acts as a pincushion for its changing properties p, represented by pins that come and go and denoted by predicates P. Some of these 'pin-properties,' however, cannot be removed without ripping the cushion apart. For they are properties that are 'essential' to s, while all other properties, those whose pins may be added or removed, are 'accidental.' We cannot take away the property of rationality from a human being without creating a freak of nature; but anyone can dye their hair a different color every day without losing their humanity.

According to this approach knowledge is firmly rooted in sense experience and one arrives at an abstract object by zooming in on only certain properties that constitute it. For example, if a basic geometrical object, like a square, is conceived of as a boundary surface of a solid die, then it does not exist independently of the die. The mind, however, can treat it as an abstract object by focusing on just the square's properties and thereby grasping the latter's form.[14] The mental processes of focussing on some aspects but neglecting others that enable the mind to take on the form of an abstract object, i.e., to identify, grasp, and know it, do not necessarily resemble the means we use to give a logical description of what it means to identify or define an abstract object. The logical reconstruction usually employs the language of abstraction. If, in the example above, the composition of the die has a number of properties $p_1, \dots, p_n$, then, by eliminating many of them (like material, color, weight, etc.), we arrive at a sub-set $p_{i_1}, \dots, p_{i_k}$ of the die's properties that characterizes an abstract object, or, in Aristotle's language, a 'secondary ousia,' like a square. We shall call this method of arriving at new objects from old ones 'Aristotelian' or 'eliminative abstraction.' In more general terms, if '$o_1[abc]$' denotes some object that has, among others, the properties a, b, c and $o_2[bcd]$ another object with properties b, c, d, then an object like $o[bc]$—that is characterized by what objects o_1 and o_2 have in common—is obtained by eliminating those properties that the two objects do not share, a and d, and possibly others. Due to

complex theory of forms (or *causae*) and how they contribute to the unity of objects, especially when two or more of them coincide, and ignore the intricate theory of how the soul, as the form of the human body, can come to know something by taking on the form of that something. We follow common practice since Boethius, though, and render 'ousia' as 'substance.'

[14]See Aristotle, *De Anima*, pt. 7. 'The so-called abstract objects the mind thinks just as, if one had thought of the snub-nosed not as snub-nosed but as hollow, one would have thought of an actuality without the flesh in which it is embodied: it is thus that the mind when it is thinking the objects of mathematics thinks as separate elements which do not exist separate. In every case the mind which is actively thinking is the objects which it thinks.'

the denotational power of language, where subjects denote substances and predicates denote properties, the process of abstraction is available in the realm of language as well. The possibility of linguistic abstraction as well as ontological abstraction has led to two different interpretations of Aristotle, but we omit further discussion of this issue.[15]

Aristotle thus established, after a heated debate that lasted for many generations and was fueled by conflicting intuitions about what the furniture of the universe is and how language can refer to it, what would eventually become a linguistic and ontological paradigm for the next two millennia. And the resulting views were not too disquieting: We are surrounded by middle-sized concrete objects whose properties are given by the senses; new objects can be obtained by eliminative abstraction, and all objects and their properties are amenable to human knowledge.[16]

The reason to call the Aristotelian paradigm 'folk ontology and semantics' is that its underlying intuitions strike most people as so natural that it requires a serious effort not think along its lines. Kant even went a step further and turned thinking according to substance and predicate from a psychological propensity into a logical necessity, i.e., made it *a priori*.[17]

Within this paradigm all concept formation is always bottom-up and well-founded; a concept cannot be legitimately formed unless each property P it contains can ultimately be traced back to some concretely given object that instantiates P or its subordinated constituents. It is this that seems to have motivated both the slogan of empiricism that nothing is in the mind that was not in the senses before and Kant's *dictum* that concepts without objects given in intuition must be empty.[18] Unsurprisingly, textbooks in the semantics of natural languages often present concepts arranged in a tree-structured hierarchy very much along the lines of Plato and Aristotle, and similar to the Porphyrian trees that emerged from that tradition.

Mathematicians appear to have embraced this approach as well whenever they proved new mathematical entities to exist by constructing them, in a bottom-up fashion, from already existing mathematical objects; like von Staudt constructed projective points as sets of 'real' points, Dedekind real numbers as sets of rational numbers, Hamilton imaginary numbers as pairs of real numbers, and so forth. (More on this in the next two sections.)

[15] See (Mueller, 1970; Lear, 1982).

[16] Recent decades have seen a revival of viable alternatives to an Aristotelian ontology based on the notion of substance, like mereology and process ontology. We shall not, however, explore their prospects in this article.

[17] By turning substance and predicate into pure concepts of the understanding and by basing a synthetic judgement *a priori*, i.e., the first analogy of experience, on the notion of substance; see (Kant, 1781, B 106, and B 244ff, resp.).

[18] See (Cranefield, 1970) on the history of the phrase *nihil est in intellectu quod non prius fuerit in sensu*, and see (Kant, 1781, B 75) for Kant's *dictum*.

The Aristotelian paradigm emerged to accommodate the needs and the language of everyday life and of sciences that hardly scratch on the surface of things; it doesn't seem to be the best choice available when it comes to understanding modern mathematics, whose development picked up incredible speed during the 19th century. We therefore wish to suggest that an approach that leaves behind the Procrustean bed of an Aristotelian ontology and the shackles of his doctrines is better suited to describe modern mathematics.

This proposal is by no means new. In particular Cassirer in his book *Substance and Function* argued for a similar point.[19] We find, however, first, his Neo-Kantian conclusions to be no longer defensible and, second, some recent accounts on structuralism to be so confused that we believe it is worthwhile to revisit the topic.[20]

We shall try to provide the evidence necessary to support our theses by way of example, for two reasons. First, a fuller scrutiny of the historical evidence would require a book-length study, something we cannot hope to accomplish within the confines of an article. Second, and much more importantly, we do not claim that the mathematical community as a whole moves (or has ever moved) like one solid block in just one direction; nothing could be farther from the truth. We would rather compare the historical development of the mathematical community with the movement of a body that various people try to pull in different directions. The vector that describes the actual movement of the body will then be the sum all those individual vectors that represent the various people. The 'vector' that describes the historical movement of the mathematical community as a whole results likewise, we suggest, from adding all individual vectors, and it points clearly, we think, in the direction of a non-Aristotelian structuralism (see Figure 1). We shall therefore be content with highlighting just a selection of those achievements that contributed more than other developments to pull mathematics towards that structuralism and readily admit that it is easy to find examples that suggest otherwise; sometimes even one and the same person can serve as a witness for both sides. While these alleged counterexamples clearly prove how diverse and vibrant the community of mathematicians has been at any given time, we also claim that their 'associated vectors'

[19]Cassirer argued for the stronger claim that all of modern science has moved away from an Aristotelian ontology of substances; see (Cassirer, 1910). Although we believe Cassirer to be basically correct about this, we have to limit our attention to mathematics.

Brendan Larvor was kind enough as to point out to the authors the work of Albert Lautman, who provided another account of the shift in cognitive style from 19th to 20th century mathematics. Since this paper is a programmatic outline only, we shall not engage in a detailed discussion here; see, however, (Larvor, 2010).

[20]See, e.g., the controversy between Hellman (2003) and Awodey (2004), or the self-inflicted difficulties Shapiro (2000) runs into when he tries to reconcile structuralism with what we called folk ontology and semantics.

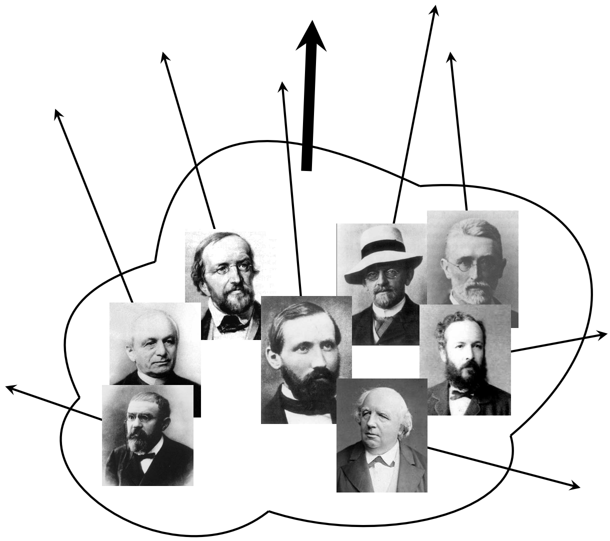

FIGURE 1. How the mathematical community *really* moves ...

have never carried weight enough to pull mainstream mathematics in their direction.

3 Towards a new ontology and semantics of mathematics

We have seen in the previous section that the notion of Aristotelian abstraction that underlies the 'folk semantics and ontology' can be interpreted both ontologically and linguistically. These two perspectives allow for the introduction of abstract objects in two different ways: in an ontological 'bottom-up' fashion and in a linguistic 'top-down' fashion.

Since our aim is not to discuss mathematics *per se*, presented in some kind of canonical form, but mathematics as a historical enterprise, its methods are certainly not fixed and have changed over time. In the following, we suggest a framework for discussing some of these developments. In particular, the move away from Aristotelian abstraction and towards 'more' abstract objects is interpreted as a move towards more 'top-down' characterizations of mathematical objects.

The main components of our framework for discussing the historical development of mathematics are illustrated in Figure 2. According to it the

$$\begin{array}{rc} & \text{Description} \\ \textit{Top-down} & \downarrow \\ & \text{Structure(s)} \\ \\ & \text{Complex objects} \\ \textit{Bottom-up} & \uparrow \\ & \text{Given objects} \end{array}$$

FIGURE 2. ‘Bottom-up’ and ‘top-down’ characterizations of mathematical objects.

introduction of mathematical objects can be achieved in two distinct ways:

I. They can be *constructed* by various means from other mathematical objects that are considered to be previously given. Historically, cuts of rational numbers and ideals have been introduced by Dedekind in this way. We refer to this approach as ‘bottom-up,’ and as will be discussed below, it is closely related to the ‘folk ontology and semantics’ paradigm mentioned above.

II. Alternatively, mathematical structures can be defined by linguistic descriptions purely in terms of their relational properties. Such definitions can be of various degrees of specificity, with ‘implicit definitions’ by systems of axioms being the most common ones (e. g., for groups and natural numbers). This ‘top-down’ approach is characteristic for modern, abstract mathematics.

The distinction between these two modes of introducing mathematical objects or structures reflects Hilbert’s distinction between the ‘genetic’ and the ‘axiomatic method’ (Hilbert, 1900). As an example of the genetic method Hilbert mentions the extension of the concept of number to include real numbers, through the successive definition of negative numbers and rational numbers as pairs, and the definition of real numbers as cuts of rational numbers. These definitions are all instances of what we call the ‘bottom-up’ approach, since the new objects are introduced as (set-theoretic) constructions on the basis of the natural numbers, which are taken as given from the outset. Hilbert’s example of the top-down, axiomatic method is Euclidean geometry, where

> one customarily begins by assuming the existence of all the elements, i. e., one postulates at the outset three systems of things (namely, the points, lines, and planes) and then [...] brings these elements into relationship with one another by means of certain axioms [...] (Hilbert 1900, p. 180; quoted from Ewald 1996, p. 1092).

$$\textit{Aristotelian abstraction}: \quad \begin{array}{c} \text{Semi-intuitive/quasi-empirical objects} \\ \uparrow \\ \text{Physical objects (world)} \end{array}$$

FIGURE 3. Aristotelian abstraction as a special case of bottom-up construction.

The notion of 'construction' employed in our description of the bottom-up approach is very general and should not be confused with the use of constructive, as opposed to classical, methods. In particular, many debates among mathematicians and philosophers, like that between Kronecker and Dedekind, are exactly about what kinds of means should be taken as legitimate for the construction of new objects. As Gray has argued, exactly such disagreements were frequently the source of anxieties that put a strain on discussions of that time (Gray, 2004). Moreover, which means are licensed by the mathematical community changed considerably during the historical development of mathematics: Cantor's and Dedekind's use of set-theoretic definitions, e.g., represented a significant extension of these means.

In general, mathematical constructions take genuine mathematical objects, like numbers, functions, or spaces as their starting point; but there is an important exception to this. A special case of this general notion of construction is *Aristotelian abstraction*, where the given objects are taken to be physical objects and the means of construction involve the deletion of particular properties of these objects (see Figure 3). Thus, according to the 'folk ontology and semantics' this particular kind of bottom-up approach is anchored in perceptible, real world objects. Such a grounding—understood either epistemically or ontologically—as tenuous as it may be, need not exist for mathematical concepts that are defined in a top-down fashion.

The determination of mathematical objects or structures by linguistic means in terms of their relations to others finds its most mature form in the implicit definitions based on systems of axioms. The axiomatic definitions of algebraic structures or the axiomatizations of various geometries are prominent examples. Drobisch's notion of 'abstraction by variation' is another example of this method of introducing mathematical concepts (Drobisch, 1875).

Despite the fact that the the top-down and bottom-up approaches are distinct in nature, in practice they are often employed side by side (see Figure 4). On the one hand, a system of objects that is constructed is often introduced with the explicit aim of satisfying particular axiomatic conditions. Hamilton's quaternions, designed to be an instance of a system in which multiplication is not commutative, and Dedekind's constructions of a simply infinite system and the system of cuts of rational numbers are good

$$\begin{array}{lcc}
 & \textit{Real numbers} & \textit{Non-commutative rings}\\
 & \text{Axioms} & \text{Axioms}\\
\textit{Top-down} & \downarrow & \downarrow\\
 & \mathbb{R}\text{-structure} & \text{Non-commutative multiplication}\\
\textit{Link} & \Updownarrow \text{ Isomorphism} & \Updownarrow \text{ Instance}\\
 & \text{Dedekind cuts} & \text{quaternions}\\
\textit{Bottom-up} & \uparrow & \uparrow\\
 & \mathbb{N} & \mathbb{N}
\end{array}$$

FIGURE 4. Examples of connecting bottom-up and top-down approaches.

examples.[21] On the other hand, axioms are often introduced to characterize a system of mathematical objects that has been constructed previously (e. g., the axiomatization of a topological space intended to capture some properties of the real line).[22] In other words, the new objects that are generated in a bottom-up fashion are often intended to instantiate a mathematical concept that has been defined using the top-down method; and vice versa, new objects that are defined in a top-down fashion are meant to be instantiated by objects that were (previously) constructed bottom-up. Through this connection the two approaches are linked and mathematicians often alternate between the two.

In our framework *mathematical work* happens in three places: In the bottom-up constructions of new mathematical objects, in developing appropriate descriptions that are the starting points for the top-down approach, and in establishing possible connections between the structures and the constructed objects (e.g., showing that they satisfy all postulated properties, or that they are even isomorphic). Both top-down and bottom-up approaches involve finding fundamental concepts and fruitful definitions, and working out their consequences.

4 Historical examples

In the following we present brief sketches of historical episodes to illustrate the point that in the development of mathematics from the 19th to the 20th century one can identify a decrease of emphasis of bottom-up characterizations and an increased reliance on top-down characterizations. Many of the historical developments that led to the emergence of modern mathematics have been presented and discussed elsewhere, and this is not the place to

[21] See the discussion of Dedekind in (Sieg and Schlimm, 2005).
[22] See (Moore, 2008).

add another such study. Instead, we adduce examples from a wide range of developments to support our claim.

The framework introduced in the previous section allows us to view seemingly very different approaches as instances of the bottom-up view. A great example is Gauss' view on the justification of new mathematical objects. Fraenkel describes it as follows:

> Gauss adopts a decidedly realist standpoint [...] according to which an extension of a given domain of numbers is only justified, if it is possible to intuitively associate with the new entities that are to be accepted other things or concepts, which have already gained general acceptance—for example, on the basis of spatial experience or spatial intuition. (Fraenkel 1920; quoted from Volkert 1986, p. 40.)

Hamilton's work on number pairs also fits right into this characterization. And the so-called 'formalism' of late 19th century mathematicians like Heine can also be understood as an instance of this general approach: In order to justify the existence of the natural numbers they are themselves reduced to something 'more concrete,' namely, written symbols. This way of proceeding was famously criticized by Frege, but he was himself working with a reductionist goal in mind (i.e., aiming at a bottom-up account), only to different kinds of objects, namely logical ones.

These views stand in stark contrast to the 'top-down' approach of Dedekind and Hilbert, who are proponents of the modern view of mathematics, according to which mathematical objects are regarded as being determined purely by their descriptions. Such an exclusive reliance on the relations expressed in axioms was a demand also formulated by Pasch—in his famous *Vorlesungen über neuere Geometrie* (1882):

> Indeed, if geometry is to be really deductive, the deduction must be independent of the [sc. bottom-up] *meaning* of geometrical concepts, just as it must be independent of the diagrams; only the *relations* specified in the propositions and definitions employed may legitimately be taken into account. (Pasch, 1882, p. 98)

While Pasch himself made this demand for the sake of gap-free deductions and did not regard mathematical objects to be defined in this way, it was soon employed as a methodological desideratum also for definitions. An early expression of this way of proceeding is given in the opening paragraph of Dedekind's *Was sind und was sollen die Zahlen?* (1888):

> In what follows, I understand by *thing* every object of our thought. In order to be able easily to speak of things, we designate them by symbols, e. g., by letters [...]. A thing is completely determined by all that can be affirmed or thought about it. (Dedekind, 1888, p. 44)

Reck refers to this as Dedekind's 'principle of determinateness' and he considers it to be a crucial component of Dedekind's 'logical structuralism' (Reck, 2003, pp. 394 & 400). A similar formulation of this principle is expressed by Hilbert:

> [...] by the set of real numbers we do not have to imagine, say, the totality of all possible laws according to which the elements of a fundamental sequence can proceed, but rather—as just described—a system of things whose mutual relations are given by the *finite and closed* system of axioms I–IV, and about which new statements are valid only if one can derive them from the axioms by means of a finite number of logical inferences. (Hilbert 1900, p. 184; quoted from Ewald 1996, p. 1095.)

A mathematical notion whose characterization has changed dramatically in the course of the 19th century is that of a function. Originally conceived as a particular, rule-based relation between numbers, it gained more and more generality in the hands of Dirichlet and Dedekind, until it was defined purely in set-theoretic terms. For a wide range of different characterizations of 'function,' see (Volkert, 1986, 55–57).

A similar development can be identified in abstract algebra, nowadays considered a prime example of axiomatically characterized structures. However, it also began as the study of concrete sets of given objects. A 'group,' for example, was defined by Galois as a set of substitutions that is closed under composition; and group-theoretic constructions were always made in terms of substitutions. Only gradually the relational structure was emphasized and taken as the essential aspect of the theory. See (Wussing, 1984) for the general development of group theory, and (Schlimm, 2008) for a particular episode that nicely illustrates our main point.

In traditional geometry, its elements were construed as abstract, usually obtained by some sort of Aristotelian abstraction. With the development of projective geometry, 'points at infinity' or 'ideal points' were introduced, but at first they were treated with the same skepticism that had been directed at the negative and imaginary numbers before. The reduction of these new geometric objects (i.e., the definition of them in terms of 'real' points and lines) was considered to be a great achievement. This sentiment is expressed, for example, in Torretti's remark on Pasch's treatment of projective geometry (Pasch, 1882):

> From a philosophical point of view, Pasch's most remarkable feat is the introduction of the ideal elements of projective geometry using only the ostensive concepts of point, segment, and flat surface and the empirically justifiable axioms S and E. (Torretti, 1978, p. 213)

Only with Hilbert's groundbreaking *Foundations of Geometry* (1899) the idea of implicit definitions of mathematical structures slowly gained general acceptance.

5 Visualizing mathematical objects

The historical transition discussed above has also had effects on the use of visualizations in mathematics, to which we turn our attention next.[23] We maintain, with Plato and Aristotle, that mathematical objects, as abstract entities, cannot be directly visualized. Euclid's definitions of a point as 'that which has no part' and of a line as a 'breadthless length' (Heath, 1909, p. 153) clearly hint at the ontological and epistemological difficulties that mathematical objects pose, but also at the problem of their accessibility to the senses. Both Plato and Aristotle agreed that these objects are not to be found in our physical world. They disagreed on the accounts of where mathematical object live and how they are related to the things we see with our eyes. Recall Aristotle's account (discussed in more detail above): the mathematical objects are idealizations of physical objects, obtained through a process of abstraction. Thus, even if there is no mathematical sphere—sometimes called 'perfect' to flag its ideal character—in the physical world, there are objects in our world that resemble such spheres to some degree. Such physical objects, imperfect instantiations as they are, can nonetheless be regarded as visualizations of their abstract counterparts. We can easily see and touch spherical objects and also imagine them; if we stretch our imagination just a little bit, we can imagine these objects to be perfectly smooth and spherical, and thus we arrive at a representation of a mathematical sphere. This representation is not identical to the sphere, but closer to it in the relevant respects than any physical object could be.[24]

In sum, some mathematical objects can be construed as idealizations of physical objects, and, accordingly, some objects are easier to visualize than others. We may refer to these as *elementary* objects, and among them we find the geometric notions of point, line, circle, square, cube, sphere, etc., the notions of natural and real numbers, and the modern notion of sets. Many philosophers have limited their discussions of the nature of mathematics to these objects and thus may have been influenced by the particular character of these kinds of objects into thinking that all of mathematics can be built up in this way.[25] However, with the increased reliance on top-down

[23] In this programatic sketch, we are unable to do justice to all the ramifications of the topic of visualization. For a more complete picture, the reader might want to look at other studies, like the contributions by Marcus Giaquinto or Ken Manders in (Mancosu, 2008).

[24] See Klein's discussion of the limits of our imagination in (Klein, 1893).

[25] The bottom-up generation of mathematical concepts via conceptual metaphors is presented in (Lakoff and Núñez, 2000).

characterizations of mathematical structures, this conception of visualization soon reaches its limits and becomes untenable as being applicable for all mathematical objects.

The relation between visual representations and mathematical objects has been a topic of debate among mathematicians themselves. In particular with the growing emphasis on rigor in the 19th century the use of diagrams was more and more scrutinized.[26] However, such representations play very different roles in mathematical practice that can be distinguished: a) Visualizations as means to mathematical *understanding* and *education*; b) visualizations as heuristics for mathematical *inferences*; c) visualizations as *justifications* of mathematical inferences; d) visualizations as vehicles for mathematical *creativity*.

Let us briefly illustrate these different roles of visualizations. Consider a formulation of Hilbert's first axiom of Euclidean geometry in the language of first-order logic, '$\forall x \forall y \exists z \ P(x) \wedge P(y) \wedge L(z) \wedge \mathrm{on}(x,z) \wedge \mathrm{on}(y,z)$.' To understand such a symbolic expression as a geometric statement the primitive terms have to be interpreted and given meaning. Reformulated into English, the statement then becomes 'between any two points there is a line.' Since the words 'points,' 'line,' and 'between' are familiar to us, we immediately understand the statement (or, at least, we think we do). Thus, the familiar terms with their associated visual representations allow us to grasp the content of a proposition much more easily. Accordingly, complex geometric propositions are often visualized using diagrams.[27] Once a proposition has been represented by a diagram, the graphical information can also be exploited for making inferences. For example, if you draw a triangle with an additional straight line going through one of its sides not at a vertex, that line, if drawn sufficiently long, will also go through one of the other two sides of the triangle. This can be easily verified using a diagram. However, diagrams can also be misleading and thus lead to incorrect proofs. Because of this, and the renewed interest in mathematical rigor in the 19th century, the tendency of rejecting the use of diagrams for licensing inferences became stronger (Mancosu, 2005).[28]

Nevertheless, it was commonly agreed that visual representations are very helpful tools in the process of forming new conjectures, since representations can suggest previously unseen connections and completely new directions of research. That multiplying and juxtaposing modes of representation leads to a 'productive ambiguity' that is crucial in the development of science and mathematics has been argued by Grosholz (2007).

[26] See (Mancosu, 2005).

[27] For a colorful example, see (Byrne, 1847).

[28] We wish to note, however, that there is danger of oversimplifying these quite complex movements within the mathematical community. Category theory, e.g., which freely and deliberately embraces diagrams and their properties (and therefore sometimes dubbed 'archery'), can serve as an antidote to such oversimplications.

In our analysis of some developments of mathematics we found a general tendency towards introducing mathematical objects in a top-down fashion, and away from the more traditional bottom-up fashion. This move correlates with a change of the role that visualizations play in mathematics. On the one hand, their justificatory power was called into question by the more urgent demands for increased rigor. On the other hand, however, visualizations became more important in their role as vehicles for promoting mathematical understanding. Since the structures defined in a top-down fashion are initially more abstract, a need was felt to provide some substance to flesh them out, and this substance was often furnished by visualizations. Examples are Klein's collections of mathematical models, or, more recently, computer visualizations of fractals. In other words, the (purely linguistic) combination of mathematical properties can lead to conceptions for which no visualization immediately springs to mind: For example, a space in which more than one parallel to a line through a given point exist, and Weierstrass' continuous, but nowhere differentiable, curves. However, mathematicians were not satisfied with this situation and put much effort into finding ways of relating these new notions to others, which were previously available, for example Beltrami's, Klein's, and Poincaré's models for non-Euclidean geometry. Thus, in general we think it is incorrect to say that the amount of and the need for visualizations has decreased in modern mathematics, but rather that the roles they play in mathematical practice have been clarified and have changed.

While bottom-up constructions of new mathematical objects were favored in the 19th century, it is characteristic for modern, 20th century mathematics to rely heavily on top-down characterizations. This focus on linguistic descriptions (axioms) went alongside the demands for more rigor in mathematical argumentations and it also provided the means for extending the limits of what is possible. For example, the simple construal of a space as $\mathbb{R}^3$ quickly led to the question of the nature of $\mathbb{R}^4$, and was generalized to $\mathbb{R}^n$, which allowed for the possibility of $\mathbb{R}^\infty$. Thus, hitherto unthinkable generalizations became possible and were being pursued. As consequences of these developments, visualizations lost their role as warrants of mathematical deductions and more abstract structures became the objects of mathematical investigations for which Aristotelian abstraction no longer works.

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