Is mathematics special?

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1 Introduction

Toutes les grandes personnes ont d'abord été des enfants
(Mais peu d'entre elles s'en souviennent.)
Antoine de Saint-Exupéry, *Le Petit Prince.*

This paper is an attempt to launch a new and somewhat unusual research programme aimed at gaining a better understanding of the specific nature of mathematical practice. I propose to systematically record and analyse logical difficulties experienced—and occasionally overcome—by children in their early learning of mathematics. Quite naturally, this limits my study to analysis of recollections of my fellow mathematicians and to interviews with so-called "mathematically able" children—only they posses an adequate language which allows them to describe their personal experiences. My approach is justified by the success of Vadim Krutetskii's (1976) classical study of mathematically able children. His book provides remarkable insights into mathematical thinking, and adult professional mathematicians instantly recognise themselves in Krutetskii's young subjects.

I am a mathematician and restrict myself to describing hidden structures of elementary mathematics which may intrigue and—like shadows in the

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Anonymous referees have helped me to understand what my paper is about and provided some very useful insights. This text would not appear without kind invitations to give a talk at the conference "Is Mathematics Special?" in Vienna in May 2008 and to give a lecture course "Elementary mathematics from the point of view of 'higher' mathematics" at Nesin Mathematics Village in Şirince, Turkey, in July 2008. The final version was prepared during my visit to the Bilgi University, Istanbul, in Spring 2009. I am grateful to all my colleagues at Bilgi and especially to fantastically hospitable staff of Santralresidence. My work on this paper was partially supported by a grant from the John Templeton Foundation. The opinions expressed in this paper are those of the author and do not necessarily reflect the views of the John Templeton Foundation.

Finally, my thanks go to the blogging community—I have picked in the blogosphere some ideas and quite a number of references—and especially to numerous (and mostly anonymous) commentators on my blog, especially to my old friend who prefers to be known only as Owl.

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night—sometimes scare an inquisitive child. I hope that my notes could be useful to specialists in mathematical education and in psychology of education. But I refrain from making any recommendations on mathematics teaching. For me, the primary aim of my project is to understand the nature of hardcore mainstream "research" mathematics.

I hope that my proposal provides an answer to the question in the title of the conference: "Is mathematics special?" Of course it is! The emphasis on child's experiences makes my programme akin to linguistic and cognitive science. However, when a linguist studies formation of speech in a child, he studies language, not the structure of linguistic as a scientific discipline. When I propose to study formation of mathematical concepts in a child, I wish to get insights into the interplay of mathematical structures in *mathematics*. Mathematics has an astonishing power of reflection, and a self-referential study of mathematics by mathematical means plays an increasingly important role within mathematical culture. I simply suggest to make a step further (or step aside, or step back in life) and take a look back in time, in one's child years.

Some very incisive comments from anonymous referees of this paper helped me to better specify its scope, and I reiterate: **(1)** I am neither a philosopher nor a psychologist. **(2)** This paper is not about philosophy of mathematics, it is about mathematics. **(3)** This paper is not about psychology of mathematics, it is about mathematics. **(4)** This paper is not about mathematical education, it is about mathematics. **(5)** But this paper has a secondary purpose: it is an attempt to trigger the chain of memories in my readers. Every, even the most minute, recollection of difficulties and paradoxes of their early mathematical experiences are most welcome.

2 Adding one by one

My colleague EHK told me about a difficulty she experienced in her first encounter with arithmetic, aged 6.[1] She could easily solve "put a number in the box" problems of the type

$$7 + \square = 12,$$

by counting how many 1's she had to add to 7 in order to get 12 but struggled with

$$\square + 6 = 11,$$

because she did not know where to start. Worse, she felt that she could not communicate her difficulty to adults. Her teacher forgot to explain to her that addition was commutative.

[1] EHK is female, English, has a PhD in Mathematics, teaches mathematics at a highly selective secondary school.

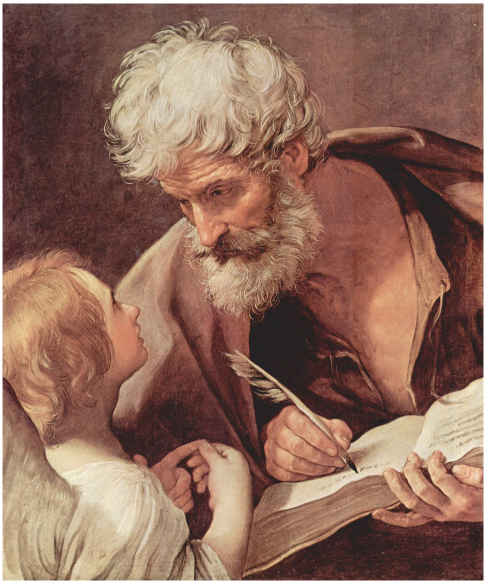


FIGURE 1. *L'Evangelista Matteo e l'Angelo.* Guido Reni, 1630–1640. Pinacoteca Vaticana. Source: *Wikipedia Commons.* Public domain. Guido Reni was one of the first artists in history of visual arts who paid attention to psychology of children. Notice how the little angel counts on his fingers the points he is sent to communicate to St. Matthew.

Another one of my colleagues, AB,[2] told me how afraid she was of subtraction. She could easily visualise subtraction of 4 from 100, say, as a stack of 100 objects; after removing 4 objects from the top, 96 are left. But what happens if we remove 4 objects from the bottom of the stack?

A brief look at axioms introduced by Dedekind (but commonly called Peano axioms) provides some insight in EHK's and AB's difficulties.

Recall that the Peano axioms describe the properties of natural numbers $\mathbb{N}$ in terms of a "successor" function $S(n)$. (There is no canonical notation for the successor function, in various books it is denoted $s(n)$, $\sigma(n)$, n', or even $n++$, as in popular computer languages C and C++.)

Axiom 1 1 is a natural number.

Axiom 2 For every natural number n, $S(n)$ is a natural number.

[2] AB is female, Turkish, has a PhD in Mathematics, teaches mathematics in a research-led university. She was 6 years old at the time of that story.

Axioms 1 and 2 define a unary representation of the natural numbers: the number 2 is $S(1)$, and, in general, any natural number n is

$$S^{n-1}(1) = S(S(\cdots S(1)\cdots)) \quad (n-1 \text{ times}).$$

As we shall soon see, the next two axioms deserve to be treated separately; they define the properties of this representation.

Axiom 3 For every natural number n other than 1, $S(n) \neq 1$. That is, there is no natural number whose successor is 1.

Axiom 4 For all natural numbers m and n, if $S(m) = S(n)$, then $m = n$. That is, S is an injection.

The final axiom (Axiom of Induction) has a very different nature and is best understood as a method of reasoning about all natural numbers.

Axiom 5 If K is a set such that 1 is in K, and for every natural number n, if n is in K, then $S(n)$ is in K, then K contains every natural number.

Thus, Peano arithmetic is a formalisation of that very counting by one that EHK did, and addition is defined in precisely the same way as EHK learned to do it: by a recursion

$$m + 1 = S(m); \quad m + S(n) = S(m+n).$$

Commutativity of addition is a non-trivial (although still accessible to a beginner) theorem. To force you to feel some sympathy to poor little EHK and to poor little AB, I reproduce *verbatim* its proof from Edmund Landau's famous book *Grundlagen der Analysis* (1930).

3 Landau's definition of addition

I use notation from Landau's (1966) book:

$$S(n) = n'.$$

Landau's proof consists of two self-contained Theorems, 4 and 6, at the very beginning of his book. Although this is not emphasised by him, the two theorems are not using Axioms 3 and 4. Landau starts by *defining* addition:

> **Theorem 4** (and at the same time Definition 1) To every pair of numbers x, y, we may assign in exactly one way a natural number, called $x + y$, such that
>
> (1) $x + 1 = x'$ for every x,

(2) $x + y' = (x + y)'$ for every x and every y.

Proof. (A) First we will show that for each fixed x there is at most one possibility of defining $x + y$ for all y in such a way that $x + 1 = x'$ and $x + y' = (x + y)'$ for every y.

Let a_y and b_y be defined for all y and be such that

$$a_1 = x', \quad b_1 = x', \quad a_{y'} = (a_y)', \quad b_{y'} = (b_y)' \text{ for every } y.$$

Let $\mathcal{M}$ be the set of all y for which

$$a_y = b_y.$$

(I) $a_1 = x' = b_1$; hence 1 belongs to $\mathcal{M}$.

(II) If y belongs to $\mathcal{M}$, then $a_y = b_y$, hence by Axiom 2,

$$(a_y)' = (b_y)',$$

therefore

$$a_{y'} = (a_y)' = (b_y)' = b_{y'},$$

so that y' belongs to $\mathcal{M}$.

Hence $\mathcal{M}$ is the set of all natural numbers; i.e., for every y we have $a_y = b_y$.

(B) Now we will show that for each x it is actually possible to define $x + y$ for all y in such a way that

$$x + 1 = x' \text{ and } x + y' = (x + y)' \text{ for every } y.$$

Let $\mathcal{M}$ be the set of all x for which this is possible (in exactly one way, by (A)).

(I) For $x = 1$, the number $x + y = y'$ is as required, since

$x + 1 = 1' = x'$,

$x + y' = (y')' = (x + y)'$.

Hence 1 belongs to $\mathcal{M}$.

(II) Let x belong to $\mathcal{M}$, so that there exists an $x + y$ for all y. Then the number $x' + y = (x + y)'$ is the required number for x', since

$$x' + 1 = (x + 1)' = (x')'$$

and

$$x' + y' = (x + y')' = ((x + y)')' = (x' + y)'.$$

Hence x' belongs to $\mathcal{M}$. Therefore $\mathcal{M}$ contains all x. Q.E.D.

It is time to pause and ask a natural question: why is the proof of *consistency* of the inductive definition of addition is so difficult? David Pierce (2009) published a timely reminder about this conundrum of the foundations of arithmetic:

> A set with an initial element and a successor-operation may admit proof by induction without admitting inductive or rather *recursive* definition of functions.

Historically, this observation was made explicit by Dedekind (1888, Remark 130) but overlooked by Peano (1889). Landau himself (1966, Preface for the Teacher) confesses to committing an error—detected by his teaching assistant—in the first version of his lectures. In a more algebraic language, the issue is clarified by Henkin (1960):

> If we consider a *unary algebra*, that is, an algebraic structure $\mathcal{N} = \langle N; 1, S\rangle$ consisting of a ground set N together with a constant symbol 1 and a unary function S (it will automatically satisfy Axioms 1 and 2), then
>
> - $\mathcal{N}$ satisfies the Axiom of Induction (Axiom 5) if and only if $\mathcal{N}$ is generated by 1;
> - $\mathcal{N}$ satisfies the Axiom of Induction (Axiom 5) together with Axioms 3 and 4 if and only if $\mathcal{N}$ is a free generated unary algebra freely generated by element 1.

Landau implicitly (and Henkin explicitly) shows that addition and —later in the book—multiplication can be defined by induction alone. But, as we have just seen, the argument takes some work.

David Pierce makes an incisive comment:

> Indeed, if one thinks that the recursive definitions of addition and multiplication—
>
> $$\begin{aligned} n+0 &= n, \\ n+(k+1) &= (n+k)+1; \\ n\cdot 0 &= 0, \\ n\cdot(k+1) &= n\cdot k+n \end{aligned}$$
>
> —are *obviously* justified by induction alone, then one may think the same for exponentiation, with
>
> $$\begin{aligned} n^0 &= 1 \\ n^{k+1} &= n^k\cdot n. \end{aligned}$$

> However, while addition and multiplication are well-defined on $\mathbb{Z}/n\mathbb{Z}$ (which admits induction), exponentiation is not; rather, we have
>
> $$\begin{array}{rcl} (x, y) & \mapsto & x^y \\ (\mathbb{Z}/n\mathbb{Z})^* \times \mathbb{Z}/\varphi(n)\mathbb{Z} & \to & \mathbb{Z}/n\mathbb{Z}, \end{array}$$

where $(\mathbb{Z}/n\mathbb{Z})^*$, as usual, denotes the group of invertible elements of the residue ring $\mathbb{Z}/n\mathbb{Z}$. Indeed, the recursive definition of exponentiation fails in $\mathbb{Z}/3\mathbb{Z}$,

n	n^2	n^3	$n^3 \times n$	n^4
2	1	2	1	2,

but holds in $\mathbb{Z}/6\mathbb{Z}$:

n	n^2	n^3	n^4	n^5	n^6	n^7
1	1	1	1	1	1	1
2	4	2	4	2	4	2
3	3	3	3	3	3	3
4	4	4	4	4	4	4
5	1	5	1	5	1	5
6	6	6	6	6	6	6

The former is an exception rather than rule, as clarified by Don Zagier's theorem.

Theorem 3.1. (Don Zagier, 1996) The identities

$$a^1 = a, \qquad a^{b+1} = a^b \times a \tag{1}$$

hold on $\mathbb{Z}/n\mathbb{Z}$ if and only if $n \in \{0, 1, 2, 6, 42, 1806\}$.

I share David Pierce's (2009) indignation at the state of affairs:

> Yet the confusion continues to be made, even in textbooks intended for students of mathematics and computer science who ought to be able to understand the distinction. Textbooks also perpetuate related confusions, such as suggestions that induction and 'strong' induction (or else the 'well-ordering principle') are logically equivalent, and that either one is sufficient to axiomatize the natural numbers. [...]
>
> This is one example to suggest that getting things straight may make a pedagogical difference.

But I have to admit that I shared the widespread ignorance until David Pierce brought my attention to the issue—despite the fact that, in a calculus course that I took in the first year of my university studies, the lecturer (Gleb Pavlovich Akilov) explicitly proved the existence of a function of natural argument defined by a recursive scheme (see Akilov and Dyatlov, 1979).

4 Induction and recursion

From a pedagogical point of view, recursion could be simpler than induction, and for two different reasons.

Firstly, recursion goes back, to smaller numbers and simpler cases. Secondly, recursion is a calculation, which is psychologically easier for children to handle than a proof. A childhood story from BB raises yet another point:[3] recursion could be more concrete than induction.

> In our math circle we covered induction (domino analogy, proofs of summation formulae such as
>
> $$1 + \cdots + n = \binom{n+1}{2},$$
>
> and varied other examples). I did passably well on the problems, but still I did not understand what the induction is really for, until the end-of-year competition. I failed to solve a single problem: arrange all binary strings of length 10 around the circle so that two adjacent differ in precisely one position (it is known as cyclic Gray code of size 10). It was when I was told the solution that I felt that I finally understood the induction. The missing element was probably the fact that I did not realize that the statement proved by induction is an honest mathematical statement that pertains to concrete numbers like 10, and not only to x, y, n, m and 1996, among which only the latter is a number, but so big and arbitrary that it could as well be denoted by n.

One of many recursive rules (the so-called "binary-reflexive" algorithm) for construction of a cyclic Gray code is obvious from the examples of codes of sizes 1, 2, 3, 4 given in Figure 2. This particular recursive algorithm is remarkable indeed for being "self-proving" (in the sense of Barry Mazur's (2007) concept of "theorems that prove themselves").

5 Landau's proof of commutativity of addition

Now we return to Landau and commutativity of addition.

> **Theorem 6** (Commutative Law of Addition)
>
> $$x + y = y + x.$$
>
> *Proof.* Fix y, and let $\mathcal{M}$ be the set of all x for which the assertion holds.

[3]BB was 11 or 12 years old at the time of the story. He is male, Russian, currently a PhD student in pure mathematics.

0	00	000	0000
1	01	001	0001
	11	011	0011
	10	010	0010
		110	0110
		111	0111
		101	0101
		100	0100
			1100
			1101
			1111
			1110
			1010
			1011
			1001
			1000

FIGURE 2.

(I) We have $y+1=y'$ and furthermore, by the construction in the proof of Theorem 4, $1+y=y'$, so that

$$1+y=y+1$$

and 1 belongs to $\mathcal{M}$.

(II) If x belongs to $\mathcal{M}$, then $x+y=y+x$, therefore

$$(x+y)'=(y+x)'=y+x'.$$

By the construction in the proof of Theorem 4, we have

$$x'+y=(x+y)'$$

hence

$$x'+y=y+x',$$

so that x' belongs to $\mathcal{M}$. The assertion therefore holds for all x.

Q.E.D.

Notice that it follows from Landau's proof that addition is defined and commutative on any 1-generated unary algebra, in particular, on the algebra shown on Figure 3.

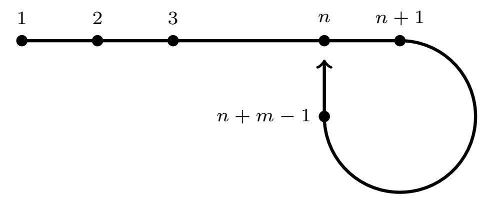


FIGURE 3. A unary algebra with commutative addition (with thanks to David Pierce).

Landau's book is characterised by a specific austere beauty of entirely formal axiomatic development, dry, cut to the bone, streamlined. Not surprisingly, it is claimed that logical austerity and precision were Landau's characteristic personal traits.[4]

Grundlagen der Analysis opens with two prefaces, one intended for the student and the other for the teacher; we already quoted *Preface for the Teacher*, it is a remarkable pedagogical document. The preface for the student is very short and begins thus:

> 1. Please don't read the preface for the teacher.
> 2. I will ask of you only the ability to read English and to think logically—no high school mathematics, and certainly no higher mathematics. [...]
> 3. Please forget everything you have learned in school; for you haven't learned it.
>
>   Please keep in mind at all times the corresponding portions of your school curriculum; for you haven't actually forgotten them.
> 4. The multiplication table will not occur in this book, not even the theorem,
>
> $$2 \times 2 = 4,$$
>
>   but I would recommend, as an exercise for Chap. I, section 4, that you define
>
> $$\begin{aligned} 2 &= 1+1, \\ 4 &= (((1+1)+1)+1), \end{aligned}$$
>
>   and then prove the theorem.

[4]Asked for a testimony to the effect that Emmy Noether was a great woman mathematician, Landau famously said: "I can testify that she is a great mathematician, but that she is a woman, I cannot swear."

Also, I would like to offer as an exercise for the reader to prove Theorem 5 from Landau's book: associativity of addition of natural numbers. It is easy. By the way, will the addition on the unary algebra of Figure 3 be associative?

Perhaps the reader would agree that little EHK and little AB had good reasons to be confused.

6 Children may take their thoughts very seriously

And here comes a really precious point communicated to me, in different words, by the two referees. I quote one of them:

> That children find start unknown problems harder than result unknown problems is well established (e.g., Riley et al., 1983) but attributing it to a lack of understanding of commutativity does not seem to be the answer: children show a grasp of commutativity in some form before they typically succeed on these problems (Canobi, 2004).

What follows is based on my earlier conversations with EHK and AB:

EHK, in her own words, did know what commutativity was; she understood that 3 apples and 2 apples was the same as 2 apples and 3 apples. She simply was not told that the same principle was applicable in her problem, because her problem was not about real life apples, it was about some entities obtained by counting one by one; an educated adult would say that the problem was about abstract objects.

AB's visualisation of subtraction as removal of objects, one by one, from the top of a stack was not about a real life stack of books, say (to start with, there were 100 of them—she distinctively remembers the concrete number in an arithmetic problem). It was a visualisation of an abstract concept of a stack (and exactly of the same stack as used in the computer science as a formalisation of push-down, "first come, first go" storage). Hers was an example of "inverse vision" in terminology of Bill Thurston (1994), visual representation of an abstract construction. She wanted to try to remove the books from the bottom of the stack, but was not sure whether she had to apply to them normal expectations of behaviour of real time objects: it did not matter for real life books where to be removed from, from the top or from the bottom, but in the second case the stack would collapse, and the stalk was important, it represented the essence of the counting algorithm.

Importantly, both EHK and AB emphasised a feeling of frustration they had because of their inability to communicate their difficulty to adults. It could be suggested, at least at a metaphorical level, that their problems were rooted in them taking the first mental structures of mathematics growing in their minds very seriously. And they do not make a representative sample—after all, both got a PhD in mathematics later in their lives.

7 "Named" numbers

I take the liberty to tell a story from my own life;[5] I believe it is relevant for the principal theme of the paper.

When, as a child, I was told by my teacher that I had to be careful with *"named"* numbers and not to *add* apples and people, I remember asking her why in that case we can *divide* apples by people:

$$10 \text{ apples} : 5 \text{ people} = 2 \text{ apples}. \tag{2}$$

Even worse: when we distribute 10 apples giving 2 apples to a person, we have

$$10 \text{ apples} : 2 \text{ apples} = 5 \text{ people} \tag{3}$$

Where do "people" on the right hand side of the equation come from? Why do "people" appear and not, say, "kids"? There were no "people" on the left hand side of the operation! How do numbers on the left hand side know the name of the number on the right hand side?

There were much deeper reasons for my discomfort. I had no bad feelings about dividing 10 apples among 5 people, but I somehow felt that the problem of deciding how many people would get apples if each was given 2 apples from the total of 10, was completely different. (My childhood experience is confirmed by experimental studies, cf. (Squire and Bryant, 2002). I suggest to call operation (2) *sharing* and (3) *dispensing* or *distribution.*)

In the first problem you have a fixed data set: 10 apples and 5 people, and you can easily visualize giving apples to the people, in rounds, one apple to a person at a time, until no apples were left. But an attempt to visualize the second problem in a similar way, as an orderly distribution of apples to a queue of people, two apples to each person, necessitated dealing with a potentially unlimited number of recipients. In horror I saw an endless line of poor wretches, each stretching out his hand, begging for his two apples. This was visualization gone astray. I was not in control of the queue! But reciting numbers, like chants, while counting *pairs* of apples, had a soothing, comforting influence on me and restored my shattered confidence in arithmetic.[6]

[5] Call me AVB; I am male, Russian, have a PhD in Mathematics, teach mathematics at a research-led university in Britain.

[6] To scare the reader into acceptance of the intrinsic difficulty of division, I refer to the paper *Division by three* (2006) by Peter Doyle and John Conway. I quote their abstract: "We prove without appeal to the Axiom of Choice that for any sets A and B, if there is a one-to-one correspondence between $3 \times A$ and $3 \times B$ then there is a one-to-one correspondence between A and B. The first such proof, due to Lindenbaum, was announced by Lindenbaum and Tarski in 1926, and subsequently 'lost'; Tarski published an alternative proof in 1949. We argue that the proof presented here follows Lindenbaum's original." Here, of course, 3 is a set of 3 elements, say, $\{0, 1, 2\}$. An exercise for the reader: prove this in a naïve set theory *with* the Axiom of Choice.

FIGURE 4. The First Law of Arithmetic: you do not add fruit and people. Giuseppe Arcimboldo, *Autumn.* 1573. Musée du Louvre, Paris.

I did not get a satisfactory answer from my teacher and only much later did I realize that the correct naming of the numbers should be

$$10 \text{ apples} \; : \; 5 \text{ people} = 2 \, \frac{\text{apples}}{\text{people}}, \; 10 \text{ apples} \; : \; 2 \, \frac{\text{apples}}{\text{people}} = 5 \text{ people}. \quad (4)$$

It is a commonplace wisdom that the development of mathematical skills in a student goes alongside the gradual expansion of the realm of numbers with which he or she works, from natural numbers to integers, then to rational, real, complex numbers:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

What is missing from this natural hierarchy is that already at the level of elementary school arithmetic children are working in a much more sophisticated structure, a graded ring

$$\mathbb{Q}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}].$$

of Laurent polynomials in n variables over $\mathbb{Q}$, where symbols $x_1, \ldots, x_n$ stand for the names of objects involved in the calculation: apples, persons, etc. This explains why educational psychologists confidently claim that the operations (1) and (2) have little in common (Squire and Bryant, 2002)—indeed, operation (2) involves operands of much more complex nature.

Usually, only Laurent monomials are interpreted as having physical (or real life) meaning. But the addition of heterogeneous quantities still makes sense and is done componentwise: if you have a lunch bag with (2 apples + 1 orange), and another bag, with (1 apple + 1 orange), together they make

$$(2 \text{ apples } + 1 \text{ orange}) + (1 \text{ apple } + 1 \text{ orange}) = (3 \text{ apples } + 2 \text{ oranges}).$$

Notice that this gives a very intuitive and straightforward approach to vectors.[7] Of course, there is no need to teach Laurent polynomials to kids; but it would not harm to teach them to teachers. I have an ally in François Viéte who in 1591 wrote in his *Introduction to the Analytic Art* that

> If one magnitude is divided by another, [the quotient] is heterogeneous to the former [...] Much of the fogginess and obscurity of the old analysts is due to their not paying attention to these [rules].

8 Digression into Turkish grammar

A logical difference between operations of sharing and dispensing is reflected in the grammar of the Turkish language by presence of a special form of numerals, *distributive numerals*.

What follows are accounts of David Pierce, Eren Mehmet Kıral and Sevan Nişanyan. David Pierce reports:

> Turkish has several systems of numerals, all based on the cardinals; as well as a few numerical peculiarities. The cardinals begin "bir, iki, üç, dört, beş, altı, ..." (one, two, three, ...) These answer the question "Kaç?" (How many?) The ordinals take the suffix -inci, adjusted for vowel harmony: "birinci, ikinci, üçüncü, dördüncü, beşinci, altıncı, ..." (first, second, third, ...). These answer the

[7]By the way, this "lunch bag" approach to vectors allows a natural introduction of duality and tensors: the total cost of a purchase of amounts g_1, g_2, g_3 of some goods at prices p^1, p^2, p^3 is a "scalar product"-type expression $\sum g_i p^i$. We see that the quantities g_i and p^i could be of completely different nature. The standard treatment of scalar (dot) product in undergraduate linear algebra usually conceals the fact that dot product is a manifestation of duality of vector spaces and creates immense difficulties in the subsequent study of tensor algebra.

> question "Kaçıncı?". The distributives take the suffix -(ş)er: "birer, ikişer, üçr, ..." Used singly, these mean "one each, two each" and so on, as in "I want two fruits from each of these baskets"; they answer the question "Kaçar?".

Eren Mehmet Kıral continues:

> When somebody is distributing some goods s/he might say "Beşer beşer alın" (each one of you take five) or "İ kişer elma alın" (take two apples each). I do not know if it is a grammatical rule (or if it is important) but when the name of the object being distributed is not mentioned then the distributive numeral is repeated as in the first example.
>
> The numeral may also be used in a non distributive problem. If somebody is asking students (or soldiers) to make rows consisting of 7 people each then s/he might say "Yedişer yedişer dizilin." (Get into rows of seven)
>
> I believe the repetition in these numerals exists to give the feeling of distribution. Since in any distribution event you give more than one time you say *beşer beşer* and indicate how many objects you are giving to the first two people, and the rest is assumed to be the same.

In that context, a story told to me by one of my colleagues, ŞUE is very interesting.[8] His experience of arithmetic in his (Turkish) elementary school, when he was about 8 or 9 years old, had a peculiar trouble spot: he could factorise numbers up to 100 before he learnt the times table, so he could instantly say that 42 factors as 6×7, but if asked, on a different occasion, what is 6×7, he could not answer. Also, he could not accept the concept of division with remainder: if a teacher asked him how many 3s go into 19 (expecting an answer: 6, and 1 is left over), little ŞUE was very uncomfortable—he knew that 3 did not go into 19. ŞUE added:

> But I did not pay attention to 19 being prime. I had the same problem when I was asked how many 3s go into 16. It is the same thing: no 3s go in 16. Simply because 3 is not a factor of 16. This is perhaps because of distributive numerals I some how built up an intuition of factorising, but perhaps for the same reason (because of the intuition that distributive gave) I could not understand division with remainder.

ŞUE came in peace with remainder only at the first year at university, when the process of division with remainder was introduced as a formal technique.

As we can see, ŞUE does not dismiss the suggestion that distributive numerals of his mother tongue could have made it easier for him to form

[8] ŞUE is Turkish, male, undergraduate mathematics student.

concept of divisibility and prime numbers (although he did not know the term “prime number”) before he learned multiplication.

9 Fractions and inductive limits

In response to my call for personal stories about difficulties in studying (early) mathematics AG sent me the following e-mail:[9]

> When I was about 9 years old, I’ve first learned at school about fractions, and understood them quite well, but I had difficulties in understanding the concept of fractions that were bigger than 1, because you see we were thought that fractions are part of something, so I could understand the concept of, for example 1/3 (you a take a piece of something you divided in 3 equal pieces and you take one), but I couldn’t understand what meant 4/3 (how can you take 4 pieces when there are only 3?). Of course I get it in several days, but I remember that I was baffled at first.

I am surprised to see how frequently such memories are related to subtle play of hidden mathematical structures, like dance of shadows in a moonlit garden; these shadows can both fascinate and scare an imaginative child. As a child, I myself was puzzled by expressions like 5/4, but my worries were resolved by pedagogical guidance: I was taught to think about fractions as named numbers of special kind: quarter apples. Fractions like 5/4 are not result of dividing 5 apples between 4 people, since this operation of division is not yet defined; they come from making sufficient number of material objects of new kind, “quarter apples” and then counting five “quarter apples”. AG was less fortunate:

> The word for fraction in Romanian is *fractie*, and that was the terminology used by my teacher and in textbooks. The word is used frequently in common language to express a part of something bigger, like fraction is used in English (I think . . .), so I think that it’s very likely that my difficulty could be of linguistic nature, I can’t eliminate also the fact that actually that was how fractions were introduce to us—pupils (like parts of an object) and only later the notion was extended, and so maybe I had problems accommodating to the new notion. Or maybe both reasons . . .

I was luckier: in Russian the word for “fraction” is reserved for arithmetic and is not used in the everyday language.[10] Also, I was clearly told to think about simple fractions $1/n$ as “named numbers” of special kind. In effect, when dealing with quarter apples, we are working in the additive semigroup $\frac{1}{4}\mathbb{N}$ generated by $\frac{1}{4}$.

[9] AG is male, Romanian, a student of Computer Science.

[10] Unless you are a hunter and use the same word for lead shot pellets.

What happens next is much more interesting and sophisticated: we have to learn how to add half apples with quarter apples. This is done, of course, by dividing each half in two quarters, which amounts to constructing a homomorphism

$$\frac{1}{2}\mathbb{N} \longrightarrow \frac{1}{4}\mathbb{N}.$$

Since both $\frac{1}{2}\mathbb{N}$ and $\frac{1}{4}\mathbb{N}$ are canonically isomorphic to $\mathbb{N}$, we, being adults now, can make a shortcut in notation and write this homomorhism simply as

$$\mathbb{N} \longrightarrow \mathbb{N}, \quad z \mapsto 2 \times z.$$

In effect, we have a direct system

$$\mathbb{N} \xrightarrow{k\times} \mathbb{N}, \quad k = 2, 3, 4, \ldots$$

—or, if you prefer less abstract notation—

$$\frac{1}{n}\mathbb{N} \xrightarrow{\mathrm{Id}} \frac{1}{kn}\mathbb{N}.$$

Then we do something outrageous: we take its inductive (or direct, which is an equivalent term) limit. In the primary school, of course, taking the inductive limit is called bringing fractions to a common denominator.

Here are formal definition of a direct set, direct system, direct (inductive) limit: A *directed set* is a nonempty set A together with a reflexive and transitive binary relation $\leq$ (that is, a *preorder*), with the additional property that every pair of elements has an upper bound.

Notice that the directed set in our definition is the set $\mathbb{N}$ with the divisibility relation. It is not linearly ordered and has a pretty sophisticated structure by itself!

Let $(I, \leq)$ be a directed set. Let $\{A_i \mid i \in I\}$ be a family of objects indexed by I and suppose we have a family of homomorphisms $f_{ij} : A_i \to A_j$ for all $i \leq j$ with the following properties: f_{ii} is the identity in A_i, $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \leq j \leq k$. Then the pair (A_i, f_{ij}) is called a *direct system* over I.

In our case, the direct system is formed by semigroups $\frac{1}{k}\mathbb{N}$, $k \in \mathbb{N}$ with natural embeddings

$$\frac{1}{n}\mathbb{N} \to \frac{1}{m}\mathbb{N} \text{ if } n \text{ divides } m.$$

The underlying set of the direct (inductive) limit, A, of the direct system (A_i, f_{ij}) is defined as the disjoint union of the A_i's modulo a certain equivalence relation $\sim$:

$$A = \bigsqcup A_i / \sim$$

Here, if x_i is in A_i and x_j is in A_j, $x_i \sim x_j$ if there is some k in I such that $f_{ik}(x_i) = f_{jk}(x_j)$.

The sum of $x_i \in A_i$ and $x_j \in A_j$ is defined as the equivalence class of $f_{ik}(x_i) + f_{jk}(x_j)$ for some k such that $i \leq k$ and $j \leq k$.

The result, of course, is the additive semigroup of positive rational numbers $\mathbb{Q}^+$. Only then we define multiplication on $\mathbb{Q}^+$. I leave a category theoretical construction of multiplication as an exercise to the reader. It is much helped by the fact that endomorphisms of the additive semigroup $\mathbb{Q}^+$ are invertible and form a group (the multiplicative group of positive rational numbers $\mathbb{Q}^\times$) which acts on $\mathbb{Q}^+$ simply transitively: for every non-zero $r, s \in \mathbb{Q}^+$ there exists a unique $q \in \mathbb{Q}^\times$ such that $s = q \cdot r$. And here is a story from BC which illustrates this point:[11]

> I could not understand the "invert and multiply" rule for dividing fractions. I could obey the rule, but *why* was multiplying by 4/3 the same as dividing by 3/4?
>
> My teachers could not explain, but I was used to that. I couldn't work it out for myself either, which was less usual.
>
> Finally I asked my father, who was an accountant. He said: if you divide everything into halves, you have twice as many things. Suddenly not just fractions but the whole of algebra made sense for the first time.

10 Palindromic decimals and palindromic polynomials

My next case study is based on conversations with an 8 year old boy, DW, in May 2007. DW's parents sent me a file of DW's book. It included the following paragraphs, reproduced here *verbatim*:

> What's weird about 1, 11, 111, 1111 etc when you square then?
>
> $1^2 = 1$. $11^2 = 121$. Keep on doing this with the other numbers. (If necessary use a calculator).
>
> Solutions see page it counts up e.g. $1111111^2 = 1234567654321$
>
> But when you have $1,111,111,111^2$ the answer is different. Figuring out (or using the calculator) when are the next square numbers in the pattern after 1111111111^2?
>
> Solutions see page 1234567900987654321, 123456790120987654321,
> 12345679012343 0987654321, 1234567901234320987654321
> and 123456790123454320987654321!
>
> Do you notice a pattern?

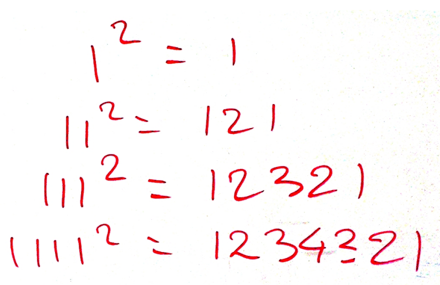
[11] BC was 10 years old. He is male, a non-English Western European.

I wrote to DW:

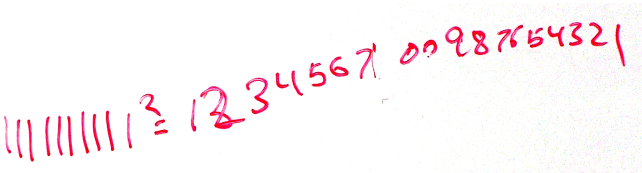
> Indeed, there is something weird. I believe you have figured out that
>
> $$\begin{aligned} 1 \times 1 &= 1 \\ 11 \times 11 &= 121 \\ 111 \times 111 &= 12321 \\ 1111 \times 1111 &= 1234321 \\ 11111 \times 11111 &= 123454321 \\ 111111 \times 111111 &= 12345654321 \\ 1111111 \times 1111111 &= 1234567654321 \\ 11111111 \times 11111111 &= 123456787654321 \\ 111111111 \times 111111111 &= 12345678987654321 \end{aligned}$$
>
> There is a wonderful palindromic pattern in the results. But mathematics is interested not so much in beautiful patterns but in reasons why the patterns cannot be extended without loss of their beauty. In our case, the pattern breaks at the next step (judging by your book, you have already noticed that):
>
> $$1111111111 \times 1111111111 = 1234567900987654321$$
>
> The result is no longer symmetric. Why? What is the difference from the previous 9 squares? Can you give any suggestions?

I had some brief e-mail exchanges with DW which suggested that he might have an explanation, but could not clearly express himself. Our discussion continued when he visited me (with his mother) in Manchester on 8 May 2007.

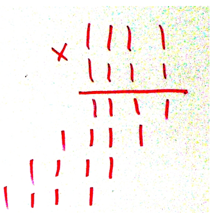
I wrote on the whiteboard in my office (the following images are photographs of actual writing on the board)



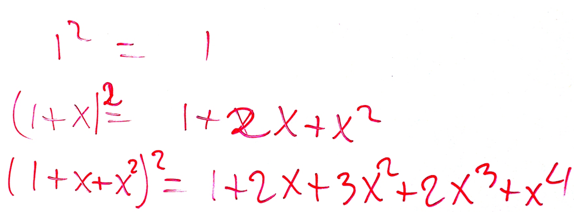
and asked DW whether the symmetric pattern of results continued indefinitely. DW instantly answered “No” and also instantly wrote on the board, apparently from his memory:



“Good,” said I, “but let us try to figure out why this is happening”, and wrote on the board:



“Yes,” said DW, “this is column multiplication”—“And what are the sums of columns’?” —“1, 2, 3, 4, 3, 2, 1,” dictated DW to me, and I wrote down the result. “Will the symmetric pattern continue indefinitely?”, I asked. “No,” was DW’s answer, “when there are 10 1’s in a column, 1 is added on the left and there is no symmetry.”—“Yes,” said I, “carries break the symmetry. But let us look at another example,” and I wrote:



DW was intrigued and made a couple of experiments (and it appeared from his behaviour that he was using mostly mental arithmetic, writing down the result, term by term, with pauses):

$$(1+x+x^2)(1+x+x^2) = 1+x+x^2+x+x^2+x^3+x^2+x^3+x^4 = x^4+2x^3+3x^2+2x+1$$

$$(1+x+x^2+x^3)^2 = 1+2x+3x^2+4x^3+3x^4+2x^5+x^6$$

Finally, he said with obvious enthusiasm: "Yes, it is the same pattern!"—"Wonderful, " I answered, "let us see why this is happening. I'll give you a hint: multiplication of polynomials can be written as column multiplication", and started to write:

$$\begin{array}{ccccccc}
 & & x^4+ & x^3+ & x^2+ & x+ & 1 \\
\cdot & & x^4+ & x^3+ & x^2+ & x+ & 1 \\
\hline
 & & x^4 & x^3 & x^2 & x & 1 \\
 & x^5 & x^4 & x^3 & x^2 & x & \\
x^6 & x^5 & x^4 & x^3 & x^2 & &
\end{array}$$

DW did not let me finish, grabbed the marker from my hand and insisted on doing it himself:

$$\begin{array}{r}
x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1 \\
\cdot\; x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1 \\
\hline
x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1 \\
x^2\;\; x\;\;
\end{array}$$

He stopped after he barely started the second line and said very firmly: "Yes, it is like with numbers"—"Well," I said, "but will the pattern break down or will continue forever?" That was the first time when DW fell in deep thought (and I was a bit uncomfortable about the degree of his concentration and retraction from the real world). This was also the first time when his response was not instantaneous—perhaps, whole 20 seconds passed in silence. Then he suddenly smiled happily and answered: "No, it will not break down!"—"Why?" I inquired. "Because when you add polynomials, the coefficients just add up, there are no carries." At that point I decided to stop the session on the pretext that it was late and the boy was perhaps tired, but, to round up the discussion, made a general comment: "You know, in mathematics polynomials are sometimes used to explain what is happening with numbers". The last word, however, belonged to DW: "Yes, 10 is x."

11 DW: a discussion

11.1 The so-called "able" children.

DW is a classical example of what is usually called a "mathematically able child". He mastered, more or less on his own, some mathematical routines—multiplication of decimals and polynomials—which are normally taught to children at much later age. He also showed instinctive interest in detecting beautiful patterns in behaviour of numbers, and, which is even more important, in limits of applicability of patterns, in their breaking points.

DW understands what generalisation is and, moreover, loves making quick, I would say recklessly quick generalisations. Krutetskii (1976) lists this trait among characteristic traits of "mathematically able" children: very frequently, they are children, who, after solving just one problem, already know how to solve *any* problem of the same type.

But let us return to the principal theme of the present paper: hidden structures of elementary mathematics. In our conversation, DW was shown—I emphasise, for the first time in his life—a beautiful but hidden connection between decimals and polynomials—*and was able to see it*!

In our little exercise, DW advanced (a tiny step) in *conceptual* understanding of mathematics: he had seen an example of how one mathematical structure (polynomials) may hide inside another mathematical structure (decimals).

My final comment is that although DW made a small, but important step towards deeper understanding of mathematics, this step is not necessarily visible in the standard mathematics education framework. It is unlikely that a school assignment will detect him making this small step. Procedurally, in this small exercise DW learned next to nothing—he multiplied numbers and polynomials before, he will multiply them with the same speed after.

One should not think, however, that the "procedural" aspect of mathematics is of no importance. DW's ability to do this tiny bit of "conceptual" mathematics would be impossible without him mastering the standard routines (in this case, column multiplication of decimals and addition and multiplication of polynomials).

11.2 Decimals and polynomials: an epiphany

DW's words "10 is x" are a formulation of analogy between decimals and polynomials which is not frequently emphasised in schools but, when discovered by children on their own, is experienced as an epiphany. These expression is taken from another childhood story, by LW.[12]

> When I was in the fourth grade (about 9 years old), we learned long division. I had enormous difficulty learning the method, though I could divide 3- and 4-digit numbers by 1- to 2-digit numbers in my head. I don't recall exactly how I did the divisions in my head, though I suspect that the method was similar to long division. I recall that it was broadly based on "seeing how much I needed to add to the result to move on." However, I couldn't seem to remember long division, despite being able to follow a list of instructions on homework. My homework on long division took hours to finish. I think the issue was that my teacher and parents never explained *why* long division actually worked, so it seemed like a disconnected list of steps that had little relation to one another. On a quiz, my teacher thought I had cheated, since I had written no work on any of the problems.
>
> I only became able to do long division in high school when I learned how to do long division with polynomials. At that time, the teacher went to great pains to explain why it worked. While doing a homework, I had an epiphany: long division for polynomials was a close cousin of long division for numbers. Suddenly I could do long division of numbers.
>
> I now take pains when teaching calculus to try and explain why formulas are true whenever possible. This has lead to mixed reactions on my evaluations, with some students saying that they enjoy knowing why the formulas are true and others saying that they have enough trouble learning formulas in the first place without having to *also* learn why they're true. For me, the two are deeply connected. If I don't know why something is true (at least in broad outline), I find it significantly harder to remember.

LW's story is an evidence in support of an observation made by Krutetskii (1976): "mathematically inclined children" may appear to be slow because they are frequently trying to solve a more general problem than the

[12] LW was 9 years at the time of his story, he is male, learned mathematics in American English (his native tongue), currently is a second year student in a PhD program in pure mathematics.

one given and understand the underlying reasons for a rule they are told to apply unquestionably.

11.3 Cohomology

But what was the mathematical object that DW dealt with? It is, of course, carry in the addition of decimals, one of the extreme cases when the deceptive simplicity of elementary school arithmetic hides a very sophisticated structure.

In Molière's *Le Bourgeois Gentilhomme*, Monsieur Jourdain was surprised to learn that he had been speaking prose all his life. I was recently reminded that, starting from my elementary school and then all my life, I was calculating 2-cocycles.

Indeed, a carry in elementary arithmetic, a digit that is transferred from one column of digits to another column of more significant digits during addition of two decimals, is defined by the rule

$$c(a,b) = \begin{cases} 1 & \text{if } a+b>9 \\ 0 & \text{otherwise} \end{cases} .$$

One can easily check that this is a 2-cocycle from $\mathbb{Z}/10\mathbb{Z}$ to $\mathbb{Z}$ and is responsible for the extension of additive groups

$$0 \longrightarrow 10\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/10\mathbb{Z} \longrightarrow 0.$$

DW discovered (without knowing the words "2-cocycle" and "cohomology") that carry is doing what cocycles frequently do: they are responsible for break of symmetry.

12 Conclusions

I can easily continue a list of my case studies. Many of them can be found in my forthcoming book (Borovik, 2010). All my examples lead to the same conclusions:

12.1 Even basic elementary mathematics is immensely rich.

I hope that this thesis is self-evident from the examples considered in the paper. For example, elementary school mathematics actually uses inductive limits and cohomology.

Although I am trying to escape from giving any advice on methods and approaches of mathematics instruction, I cannot avoid making a few general remarks.

First of all, glossing over difficulties presented by hidden structures may seriously imperil students' progress.

Next, it is desirable (but perhaps not always realistic) that teachers are aware about the hidden structures and are able to guide pupils around dangerous spots—perhaps without needlessly alerting them every time.

12.2 Hidden structures: obstacles or opportunities?

Another preliminary conclusion of this study is that the sensitivity to the presence of "hidden" structures appears to be an important component of mathematical ability in so-called "mathematically gifted" children. Special forms of this observation can be found already in the classical study by Vadim Krutetskii (1976) who pinpoints numerous instances of children using "symmetry" considerations or intentionally resorting to more general arguments.

And this is point where a reminder about non-representativity of the sample of respondents is necessary. One of the respondents (BB, quoted in Section 4) formulated this caveat very explicitly:

> I would like to point out that the stories are provided by the people who are [...] atypical in terms of mathematical thinking and learning. Most people who have responded have not only gone to deal with unusually abstract concepts in their career, but actually do mathematics. So, the examples here might represent not so much the major difficulties that need to be overcome (as in finding the correct way of thinking of division apples by apples) before an understanding can be reached, but the signs that the understanding has already been reached, and difficulty is purely semantic, i.e., how to express it.

Perhaps, stories told in this paper represented difficulties for the teacher and missed opportunities for the child.

12.3 Intrinsic logical structures of human languages

We had a chance to see in several stories that children's perception of mathematics can be affected by logical and mathematical structures (such as systems of numerals) of the language of mathematics instruction and their mother tongue. Intrinsic structures of natural languages are engaged in a delicate interplay with hidden structures of mathematics.

For me personally, this is a serious practical issue. Every autumn, I teach a foundation year (that is, zero level) mathematics course to a large class of students which includes 70 foreign students from countries ranging from Afghanistan to Zambia. Students in the course come from a wide variety of socioeconomic, cultural, educational and linguistic backgrounds. But what matters in the context of the present paper are invisible differences in the logical structure of my students' mother tongues which may have huge impact on perception of mathematics. For example, the connective "or" is strictly exclusive in Chinese: "one or another but not both", while in English "or" is mostly inclusive: "one or another or perhaps both". Meanwhile, in mathematics "or" is always inclusive and corresponds to the expression "and/or" of bureaucratic slang. In Croatian, there are two connectives "and": one *parallel*, to link verbs for actions executed simultaneously, and another *consecutive*.[13]

But it is as soon as you approach definite and indefinite articles that you get in a real linguistic quagmire. In words of my correspondent VČ:[14]

> [In Croatian, there are] no articles. There are many words that can "serve" as indefinite articles (*neki*=some, for example), but not no particularly suitable word to serve as definite article (except adjective *određeni* = definite, I guess). Many times when speaking mathematics, I (in desperation) used English articles to convey meaning (eg. *Misliš da si našao* ***a*** *metodu ili* ***the*** *metodu za rješavanje problema tog tipa? = You mean you found* ***a*** *method or* ***the*** *method for solving problems of that type*?

Psycholinguistic aspects of learning mathematics are a deserving area of further study.

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[13] Rudiments of "consecutive and" can be found in my native Russian and traced to the same ancient Slavic origins.

[14] VČ is male, Croatian, a lecturer in mathematical logic and computer science.

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