Contradictions in Mathematics

Manuel Bremer

Philosophisches Institut, Heinrich-Heine-Universität Düsseldorf, Universitätsstraße 1, 40225 Düsseldorf, Germany

E-mail: `bremer@mbph.de`

Contradictions are typically seen as anathema to mathematics. As formalism sees consistency as the only condition to consider some mathematical structure as an object of study, inconsistency becomes the single excluding criterion. For mathematical realists accepting inconsistencies comes down to accepting inconsistent objects, which just seems bizarre.

In this paper I consider two inconsistency friendly approaches in (the philosophy of) mathematics. In a recent study *How Mathematicians Think*, William Byers (2007) argues that one way of mathematical progress is by way of contradiction. The first paragraph outlines Byers' thesis, but it turns out that contradictions play a role only *ex negativo*. In contrast to that the approach of inconsistent mathematics claims contradictions to be real. Especially in inconsistent arithmetic contradictions are said to play a vital role. They turn out to provide a framework for a finitist position which endorses inconsistent numbers.

1 Byers' Creative Use of Contradictions

William Byers (2007) claims mathematics to be at core a creative activity. Mathematical reasoning, according to Byers, is not primarily algorithmic or based on proof systems, but is based on using (great) "ideas" to shed new light on mathematical objects and structures. These ideas not only are placed at the centre of mathematical understanding, which Byers calls "turning on the light", but also propel mathematical progress. Byers presents a couple of examples in which a crucial step forward in the development of mathematics depended on the presence of two at first sight unrelated or even barely compatible perspectives on some mathematical structure. He starts with the discovery of the irrational numbers (like $\sqrt{2}$), where $\sqrt{2}$ is clearly present as a geometric object (the length of the hypothenuse of the right angled triangle with unit length sides) but is not allowed for by (early Greek) arithmetic. The real numbers "provide a context" (p. 38) in which the two perspectives are unified. Another famous example is the *Fundamental Theorem* of the calculus, which says "that there is in fact one process in calculus that is integration when it is looked at in one way and differentiation when it is looked at in another" (p. 50). The core of mathematics, according to Byers, is finding such situations and being able to understand them by providing a more comprehensive view. This process is creative and

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not algorithmic. Proofs only sum up the discovery and preserve the results in text books. Mechanical proofs Byers sees as "trivial" (p. 373) whereas "deep" proofs are framed in expressing some (great) "idea" (like re-ordering infinite series makes it obvious to see a sum formula). Some proofs (like diagonalization with Cantor's original insights) are part of discovery, but these are rather the exception than the rule.

Good mathematicians are, therefore, those who hit on "ideas" (like Cantor hitting on diagonalization and the continuum hypotheses). Even more revolutionary are "great ideas". An example of a great idea is formalism. Formalism provided a unifying perspective on the whole of mathematics. When Hilbert started with formalizing Euclid's geometry "*formalism* was born and, in the process, the whole notion of truth was radically transformed" (p. 291). A great idea is then inflated (like in Hilbert's claims on behalf of formalism) and then again delimited in a wider perspective (like when Gödel's Theorems hit formalism). As ideas are outbursts of creativity "the answer to the question of whether a computer could ever do mathematics is clearly 'No!' " (p. 369). Byers finally relates his view to the question of how mathematics is to be taught, namely by getting students understand the ideas to "turn on the light".

One of the central methodological concepts – besides ambiguity – Byers uses in analysing the examples he presents is 'contradiction'. The very subtitle of his book reads 'Using Ambiguity, Contradiction, and Paradox to Create Mathematics'.

'Contradiction' is understood by Byers in two ways. On the one hand we have two seemingly contradictory perspectives in some of the mathematical problems he presents. For example, one may see $\sqrt{2}$ as a decimal, "an 'infinite' indefinite object" (p. 97), but also as a finite geometric object. One can see "2 + 3 = 5" both as expressing a fact of identity (i.e., something static) as well as expressing the process of adding (i.e., something dynamic). But of course the fact can be established by going through the process of adding, the two perspectives are finally compatible and not inconsistent. The paradox of zero (as something that is nothing) vanishes with axiomatization.

On the other hand some seminal proofs work by using contradictions, or so Byers claims. For a simple example, one can argue for the proposition that a straight line falling on parallel straight lines makes the alternate angles equal to one another in the following fashion:

> Suppose on the contrary, that angle α is not equal to angle β, for example, angle α is smaller than angle β. Then by adding angle γ we end up with
>
> $$\text{angle } \alpha + \text{angle } \gamma < \text{angle } \beta + \text{angle } \gamma = \text{two right angles.}$$

> Thus the interior angles on the same side are less than two right angles. The parallel postulate tells us that in this situation the lines must meet, contradicting the assumption that they are parallel. (p. 95)

Other famous examples are Cantor's use of diagonalization or Gödel's Theorems.

Now, if we look at these examples it becomes obvious that none of the mathematicians in question endorses any of the contradictions. Quite the opposite. In these indirect proofs contradictions are used as a threat to establish the opposite result. Just for reduction some innocent looking assumption is made which turns out to be contradictory and thus untenable. What we really see here is not a creative use of contradictions, but the creative use of indirect proof methods. Mathematics still avoids the contradictory. Even supposedly incompatible perspectives on one and the same structure have to be kept distinct from contradictions. The paradoxical calls for resolution. The perceived incompatibility is the very reason to look for another solution. At last Byers admits "we cannot leave it at that—things *must* be reconciled" (p. 111). "New stages of mathematical development arise out of a 'resolution' of a set of paradoxes" (p. 186).

2 Inconsistent Mathematics and Finitism

To have *an inconsistent number theory* means at least that within the theorems of number theory there is some sentence A with A being a theorem and $\neg A$ being a theorem at the same time. Supposedly this contradiction corresponds to some object/number a being an inconsistent object. So inconsistent mathematics is connected to inconsistent ontology. Its underlying logic has to be paraconsistent (cf. Bremer, 2005). Changing the basic logic used in mathematics to a paraconsistent logic makes mathematics in a weak sense paraconsistent: If there were to turn up some inconsistency in mathematics, it would not explode. Explosion would happen in standard logic as one can derive any sentence whatsoever from a contradictions, as it implies everything; this move typically is blocked in paraconsistent logics.

The problems with having $F(a)$ and $\neg F(a)$ for some object a seem less pressing if a is some mathematical object than if a is a physical object. Mathematical objects are either non-existent—*mere* theory, taken instrumentally—or they are in some elusive Platonic realm where strange things may well happen. If on the other hand one is a reductionist realist about mathematics (mathematics being about structures of reality or mathematical entities rather being concrete entities dealt with by mereology) then inconsistent mathematics is as problematic as your cat being (wholly) black and not being (wholly) black at the same time.

Philosophers when concerned with mathematics focus on number theory, since the ontological questions of mathematics ("What and where are mathematical objects?", "Are there infinite sets?", ...) and the epistemological questions of mathematics ("How do we know of numbers?", "Is mathematics merely conventional?", ...) do arise already with number theory.

Taking set and model theory as part of logic anyway, logicians are also mainly concerned with number, since a lot of meta-logical theorems make us of the device of arithmetization.

The same goes for the general theory of automata and computability. I follow this focus here and so this section of the paper concerns itself mostly with arithmetic. This may not be enough for a mathematician trying to assess the power of inconsistent mathematics. She looks for inconsistent theories at least of the power of the calculus. There are actually such theories, e.g., presented by Chris Mortensen (1995).

One of the most fundamental mathematical theories is arithmetic (as given for instance by the Dedekind/Peano axioms). We are here concerned mainly with first order representations. In distinction to an axiomatic arithmetic theory like Peano Arithmetic there is the arithmetic N (being the set of true first order arithmetic sentences in the standard interpretation). N is negation complete (either A or $\neg A$ is in N), not axiomatisable, not decidable, and, of course, infinitely large.

Given its first order representation there are a lot of well-known theorems about arithmetic (e.g., Peano Arithmetic being negation- and ω-incomplete).

Using the compactness theorem for first order logic, one can prove that there are non-standard models of Peano Arithmetic, which contain additional numbers over and above the natural numbers. These additional numbers behave consistently, however. Consistency provides them in the first place. Inconsistent arithmetic may concern itself with the opposite deviance: Having arithmetics where there are less numbers than in standard arithmetic.

This is of utmost philosophical interest, since the infinite is a really problematic concept leading to the ever larger cardinalities of "Cantor's paradise", and finitism (in the sense of the assumption that there are only *finitely many* objects, even of mathematics) is therefore an option worth exploring and pursuing.

Robert Meyer (1976) was the first to give a non-triviality proof of a Relevant (paraconsistent) arithmetic. The system $\mathbf{R}^{\#}$ is an extension of the first order version of Relevant logic $\mathbf{R}$ with axioms mirroring those of Peano Arithmetic save that the "$\supset$" in them has been replaced by the Relevant "$\rightarrow$". Induction is present as a rule. $\mathbf{R}^{\#}$ is non-trivial in that $0 = 1$ is not provable. This non-triviality can be established by finitistic methods.

Inconsistent arithmetics that are finite (in their models) may have any finite size you like. They contain one largest number. A largest number is a very special number indeed: usually it is understood as verifying a statement as to the identity of that number to its (supposed) successor. One may object that this is difficult to conceive. But as often with mathematical or scientific claims our capacities for (visual) imagery may not have the last word here. We can characterise a largest number as that item which fulfils certain axioms or theorems. Seen thus a largest number is an entity postulated by a (mathematical) theory like, say, a large cardinal.

Since we do not know which number really is the largest we may assume that one of these finitistic arithmetics is true, although we don't know which. Which one it is is not that important, since all these arithmetics have common properties:

We can define a sequence N_n of finite, inconsistent arithmetics with the following properties (cf. Priest, 1994a,b, 1997):

(i) For every $n \in \mathbb{N}$, we have $\mathsf{N} \subset \mathsf{N}_n$.

(ii) For every $n \in \mathbb{N}$, N_n is inconsistent.

(iii) For every $n \in \mathbb{N}$, if A is a (negated) equation for numbers $< n$, then $A \in \mathsf{N}$ if and only if $A \in \mathsf{N}_n$.

(iv) Every N_n is decidable.

(v) For every $n \in \mathbb{N}$, N_n is representable in N_n (i.e., we have a truth predicate for N_n).

(vi) For the proof predicate B of N_n, every instance of $\mathrm{B}(\ulcorner A \urcorner) \supset A$ is in N_n.

(vii) If A is not a theorem of N_n, then $\neg\mathrm{B}(\ulcorner A \urcorner)$ is in N_n.

(viii) If G_n is the Gödel sentence for N_n, then both G_n and $\neg\mathrm{G}_n$ are in N_n.

These inconsistent arithmetics N_n thus have quite remarkable properties: The theories N_n are negation complete (by (i)) and inconsistent (by (ii) and (viii)). By (iv), they have all the nice properties that N does not have, though the N_n are complete. By (v), we can define a truth predicate in the language of arithmetic for the same language, and by (vi), N_n has an ordinary proof predicate, using a standard Gödel numbering if enough numbers are available. Finally, by by (vii) in conjunction with (iii), we have not only that N_n is non-trivial (by excluding some the equations that are excluded by N), but that this non-triviality can be established within N_n itself.

Let us give a few comments about the proof of the properties of the theories N_n: First of all, a theory with fewer numbers has fewer counterexamples to a given arithmetic sentence. Thus, more sentences are true (in general, this is called the "collapsing lemma"). Since N is negation complete, any properly stronger theory will have to add a sentence whose negation is already in N. Thus, for at least one A, the resulting theory must contain both A and $\neg A$. This means that the logic of these arithmetic theories has to be a paraconsistent logic.

Representability of truth is a consequence of (iv) and (i). The same holds for the representability of the proof predicate, (vi). Once the proof predicate is representable in the decidable theory N_n, we can represent non-provability, and thus have (vii) and finally (viii).

The most interesting property is (iii) which results from the way the domain of a corresponding model is constructed.

A model of a theory N_n is constructed as a filtering of an ordinary arithmetic model. In general one can reduce the cardinality of some domain by substituting for the objects equivalence classes given some equivalence relation. The equivalence classes provide then the substitute objects. Since the objects within the equivalence class are equivalent in the sense of interest in the given context the predicates still apply (now to the substitute object). The trick in case of N_n is to chose the filtering which puts every number $< n$ into its equivalence class, and nothing else; and puts all numbers $\geq n$ into n's equivalence class. As a result of this for $x < n$ the standard equations are true (of $[x]$), while in case of $y \geq n$, *everything* that could be said of such a y is true of $[n]$. So we have immediately $n = n$ (by identity) and $n = n + 1$ (since for $y = n + 1$ in N this is true). So, the domain of a theory N_n is of cardinality n. The number n becomes an inconsistent object of N_n. Drawing the successor function by arrows, the structure of a model of N_n looks like this:

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n \circlearrowright$$

These models are called "heap models". The logic of N_n has to be paraconsistent. And it has to have restrictions on standard first order reasoning as well.

Mortensen chooses **RM3#** as basic system and finitizes it by substituting for a number n the number n modulo some m. Thus the domain becomes $\{0, 1, 2, \ldots, m-1\}$. The models then are no longer heap models but *circular* of size m. The resulting arithmetic $\mathbf{RM3}^m$ is negation complete, non-trivial and decidable. $\mathbf{RM3}^m$ is *axiomatisable* by adding to **RM3#** the axioms:

$$\vdash 0 = m$$

and all instances of the following axiom scheme for $n \in \{0, 1, \ldots, m-1\}$:

$$\vdash (0 = n \leftrightarrow 0 = 1).$$

The approach "modulo some m" has at least the same deviant results than the heap models mentioned before: In $\mathbf{RM3}^5$ we have $4+2 = 6$ and $4 \times 6 = 4$; this approach gets deviant sentences for some *known* numbers!

Arithmetic is constructed as a finite theory. One can generalize the steps of this procedure to apply it to other mathematical theories. van Bendegem (1993) distinguishes the following steps:

(i) Take any first-order theory T with finitely many predicates. Let M be a model of T.

(ii) Reformulate the semantics of T in a paraconsistenf fashion (i.e., the mapping to truth values and overlapping extensions of P^+ and P^-).

(iii) If the models of M are infinite, define an equivalence relation R over the domain D of M such that D/R is finite.

(iv) The model M/R is a finite paraconsistent model of the given first-order theory T such that validity is at least preserved.

The restriction to theories with finitely many predicates is no real restriction in any field of applied mathematics or formal linguistics, since no physical device (be it human or machine) can store a non-enumerable list of basic predicates. Van Bendegem hints at finite version of the theory of integers and the theory of rational numbers. Mortensen (1995) considers some inconsistent version of the calculus.

The *Löwenheim/Skolem-Theorem* is one of the limitative or negative meta-theorems of standard arithmetic and first-order logic. It says that any theory presented in first-order logic has a *denumerable* model. This is strange, since there are first order representations not only of real number theory (the real numbers being presented there as uncountable/non-denumerable), but of set theory itself. Thus the denumerable models are deviant models (usually Herbrand models of self-representation), but they cannot be excluded. Given the general procedure to finitize an existing mathematical first order theory using paraconsistent semantics, there is a paraconsistent strengthened version of the *Löwenheim/Skolem-Theorem*:

Theorem 2.1. Any mathematical theory presented in first order logic has a *finite* paraconsistent model.

3 The Benefits of Inconsistency

A mathematics that does not commit us to the infinite is a nice thing for anyone with reductionist and/or realist leanings. As far as we know the universe is finite, and if space-time is (quantum) discrete there isn't even an infinity of space-time points. The largest number may be indefinitely large. So we never get to it (e.g., given our limited resources to produce numerals by writing strokes). If there is a largest number n there is the corresponding inconsistent arithmetic N_n. We can presuppose N_n being our arithmetic. Since N and N_n agree on all finite and computational mathematics it is hard to see whether we lose anything important at all by switching to N_n. If we have paraconsistency anyway for other reasons, we get this finitism for free, it seems. So why not take it? In as much as N_n is correct no correct reasoning transcends the finite. Hilbert wouldn't have rejoiced, probably, since N_n of course is inconsistent itself. The drawback of all this is, of course, the problem of an ontology of inconsistent entities—at least if you are a realist.

If there are inconsistent versions of more elaborated mathematical fields like the calculus one may draw some general philosophical conclusions: Firstly, if there are corresponding inconsistent versions of these mathematical theories with comparable strength to the original theories then consistency is not the fundamental mathematical concept, but functionality (of the respective basic concepts) may well be.

Moreover, if the justification of mathematics depends on its applicability and the inconsistent versions are of comparable applicability then they are justified not just as mathematical theories, but even in the wider perspective of grasping fundamental structures of reality; there no longer will be available the argument from mathematical describability to the consistency of the world.

One final worry may be, which contradictions to accept. If we (mis-)understood inconsistent arithmetic as allowing for any old contradictions, this would be the end of all serious study, wouldn't it? Roughly the inconsistent mathematician accepts only those contradictions that are forced on her by one of the following two commitments:

The first is a commitment to some principles which are intuitively valid and superior to their circumscribed rivals, like Naïve Comprehension in set theory. Going into the details here would lead us to the wider debate surrounding dialetheism and paraconsistency, but dialetheist typically argue that the supposed solutions to the antinomies like a semantic or set theoretic hierarchy are even more mysterious than dialetheism and additionally violate our intuitive conceptions, of sets for example (cf. Bremer, 2005, pp. 20–31). The accepted contradictions are then these which follow (more or less immediately) from the presence of the accepted principles like Naïve Comprehension (e.g., the Russell Set being a member of itself and not being a member of itself). These contradictions are isolated, however, as a paraconsistent logic bars the spreading of contradictions.

The second is the commitment to a finitist universe. The broader picture of the universe one might like to endorse may be that of a finite universe without additional abstract entities. This is a highly controversial metaphysical agenda, but one that because of its naturalness should be taken as a serious rival to the more ontologically promiscuous standard picture of infinite Platonism (cf. Bremer, 2007). The accepted contradictions are then those which immediately concern the largest, and therefore inconsistent, number(s). Again the presence of a paraconsistent logic prevents these contradictions from spreading.

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